

WEIGHTED LOW-REGULARITY SOLUTIONS OF THE KP-I INITIAL-VALUE PROBLEM

J. COLLIANDER, A. D. IONESCU, C. E. KENIG, AND G. STAFFILANI

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1. INTRODUCTION

In this paper we consider the KP-I initial-value problem

$$\begin{cases} \partial_t u + \partial_x^3 u - \partial_x^{-1} \partial_y^2 u + \partial_x(u^2/2) = 0; \\ u(0) = \phi, \end{cases} \quad (1.1)$$

on $\mathbb{R}_{x,y}^2 \times \mathbb{R}_t$. The dispersion function for this dispersive equation is for $(\xi, \mu) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$

$$\omega(\xi, \mu) = \xi^3 + \mu^2/\xi.$$

In [2] three of the authors studied (1.1) with initial data ϕ in the space $E \cap P$ defined below. (See the introduction and the references of [2] for a discussion of (1.1), its relationship to the corresponding IVP for the KP-II equation, and a

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discussion of the spaces E and P in connection with (1.1)). The main result in [2] is a weak form of local in time well-posedness (Theorem 1 in [2]) for data which is small in $E \cap P$. Unfortunately, A. Ionescu discovered a counterexample to the main estimate used in [2] (Theorem 3 in [2]) to establish Theorem 1. The example exhibits a logarithmic divergence in the estimate, which shows that the proof of Theorem 1 in [2] is incorrect. The same applies to Theorem 2 in [2]. The counterexample is explained in subsection 1.1 below. Colliander, Kenig and Staffilani are very grateful to Ionescu for pointing out this mistake and for joining them in this work. Here we obtain a strengthening of Theorem 1 in [2] which yields the strong form of local in time well-posedness for small data in $E \cap P$. This is Theorem 1.1 below. The logarithmic divergence is avoided by introducing new resolution spaces, inspired by those used by Ionescu-Kenig ([3, 4, 5]) in works on Benjamin-Ono equation and on the Schrödinger map problems. It seems very likely that using the tools developed here, a correct (and similarly strengthened) version of Theorem 2 in [2] could also be obtained. We have felt, however, that this would increase substantially the technicalities in an already very technical paper and we have therefore not pursued this issue.

We conclude by mentioning that our main theorem does not give local well-posedness in $E \cap P$ for large data; such a result would immediately yield global in time well-posedness.

1.1. The counterexample. We start this section with some notation and by recalling some spaces of functions introduced in [2]. We denote the Fourier transform of a function $f(x, y)$ as

$$\hat{f}(\xi, \mu) = \mathcal{F}f(\xi, \mu) = \int_{\mathbb{R}^2} f(x, y) e^{i\langle (x, y), (\xi, \mu) \rangle} dx dy. \quad (1.2)$$

Now let χ_A denote a smooth characteristic function of the set A .

Definition. Let $\theta_0(s) = \chi_{[-1, 1]}(s)$, $\theta_m(s) = \chi_{[2^{m-1}, 2^m]}(|s|)$, $m \in \mathbb{N}$. For $(\xi, \mu) \in \mathbb{R}^2$ let $\chi_1(\xi, \mu) = \chi_{\{|\xi| \geq \frac{1}{2} \frac{|\mu|}{|\xi|}\}}$, and $\chi_2(\xi, \mu) = \chi_{\{|\xi| < \frac{1}{2} \frac{|\mu|}{|\xi|}\}}$. Let $\chi_0(s) = \chi_{\{|s| < 1\}}$, $\chi_j(s) = \chi_{\{2^{j-1} \leq |s| < 2^j\}}$ and $w(\xi, \mu) = (1 + |\xi| + |\mu|/|\xi|)$. We define the space $X_{s,b}$ through the norm

$$\begin{aligned} \|f\|_{X_{s,b}} &= \sum_{j,m \geq 0} 2^{jb} \left(\int_{\mathbb{R}^3} \chi_j(\tau - \omega(\xi, \mu)) \chi_1(\xi, \mu) \theta_m(\xi) w^{2s} |\hat{f}|^2(\xi, \mu, \tau) d\xi d\mu d\tau \right)^{\frac{1}{2}} \\ &+ \sum_{j,m \geq 0} 2^{jb} \left(\int_{\mathbb{R}^3} \chi_j(\tau - \omega(\xi, \mu)) \chi_2(\xi, \mu) \theta_n(\mu) w^{2s} |\hat{f}|^2(\xi, \mu, \tau) d\xi d\mu d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

We also define the space

$$Y_{s,r,b} = \{f : tf \in X_{s,b}, \text{ and } yf \in X_{r,b}\},$$

and the spaces

$$Z_{s,b} = X_{s,b} \cap Y_{s,1-s,b}, \quad \text{and } Z_{1-\epsilon} = Z_{1-\epsilon, \frac{1}{2}}.$$

We recall here the statement of Theorem 3 in [2]:

Theorem. *Assume $0 < \epsilon_0 < \frac{1}{8}$. Then for any $\frac{1}{4} < \epsilon < 1$, we have*

$$\begin{aligned} \|\partial_x(uv)\|_{X_{1-\epsilon_0, -\frac{1}{2}}} &\leq C\|u\|_{X_{1-\epsilon_0, \frac{1}{2}}}(\|v\|_{X_{1-\epsilon_0, \frac{1}{2}}} + \|v\|_{X_{1-\epsilon_0, \frac{1}{2}}}^{1-\epsilon} \|v\|_{Y_{1-\epsilon_0, -\epsilon_0, \frac{1}{2}}}^\epsilon) \\ &\quad + C\|v\|_{X_{1-\epsilon_0, \frac{1}{2}}}(\|u\|_{X_{1-\epsilon_0, \frac{1}{2}}} + \|u\|_{X_{1-\epsilon_0, \frac{1}{2}}}^{1-\epsilon} \|u\|_{Y_{1-\epsilon_0, -\epsilon_0, \frac{1}{2}}}^\epsilon) \end{aligned} \quad (1.3)$$

This theorem unfortunately cannot hold since the following counterexample shows a logarithmic divergence. Let $\psi : \mathbb{R} \rightarrow [0, 1]$ denote a smooth function supported in the interval $[-2, 2]$ and equal to 1 in the interval $[-1, 1]$. Assume $N \gg 1$ is very large, $\omega(\xi, \mu) = \xi^3 + \mu^2/\xi$, and define

$$\widehat{u}(\xi, \mu, \tau) = \psi(\xi - N)\psi((\mu - \sqrt{3}\xi^2)/\xi)\psi(\tau - \omega(\xi, \mu)), \quad (1.4)$$

and

$$\widehat{v}(\xi, \mu, \tau) = \psi(\xi - 4)\psi((\mu + \sqrt{3}\xi^2)/\xi)\psi(\tau - \omega(\xi, \mu)). \quad (1.5)$$

Notice that in the definition (1.4) $|\mu - \sqrt{3}N^2| \leq CN$ and in the definition (1.5) the variable ξ is about 1 (bounded away from 0). The functions $\mu \rightarrow \psi((\mu - \sqrt{3}\xi^2)/\xi)$ in (1.4) and $\mu \rightarrow \psi((\mu + \sqrt{3}\xi^2)/\xi)$ in (1.5) are essentially the characteristic functions of the intervals $[\sqrt{3}N^2 - N, \sqrt{3}N^2 + N]$ and $[-16\sqrt{3} - 1, -16\sqrt{3} + 1]$ respectively. The precise formulas $\psi((\mu - \sqrt{3}\xi^2)/\xi)$ and $\psi((\mu + \sqrt{3}\xi^2)/\xi)$ are convenient for the nonlinear change of variables (1.14). Then

$$\begin{aligned} \|u\|_{X_{1-\epsilon_0, 1/2}} &\approx N^{1-\epsilon_0} N^{1/2}, \quad \|u\|_{Y_{1-\epsilon_0, -\epsilon_0, 1/2}} \approx N^{1-\epsilon_0} N^{1/2}, \\ \|v\|_{X_{1-\epsilon_0, 1/2}} &\approx 1, \quad \|v\|_{Y_{1-\epsilon_0, -\epsilon_0, 1/2}} \approx 1. \end{aligned} \quad (1.6)$$

So the right-hand side in (1.3) is

$$RHS \approx N^{1-\epsilon_0} N^{1/2}. \quad (1.7)$$

We look now at the left-hand side of (1.3): the function $\widehat{u} * \widehat{v}$ is supported in the set $\{(\xi, \mu, \tau) : |\xi - N| \leq C \text{ and } |\mu - \sqrt{3}N^2| \leq CN\}$. So,

$$\|\partial_x(uv)\|_{X_{1-\epsilon_0, -1/2}} \approx N \cdot N^{1-\epsilon_0} \sum_{j \geq 0} 2^{-j/2} \|(\widehat{u} * \widehat{v})(\xi, \mu, \tau) \chi_j(\tau - \omega(\xi, \mu))\|_{L_{\xi, \mu, \tau}^2},$$

where χ_j is the characteristic function of the set $\{s : |s| \in [2^{j-1}, 2^{j+1}]\}$. Using (1.7), it would follow from (1.3) that

$$\sum_{j \geq 0} 2^{-j/2} \|(\widehat{u} * \widehat{v})(\xi, \mu, \tau) \chi_j(\tau - \omega(\xi, \mu))\|_{L_{\xi, \mu, \tau}^2} \leq CN^{-1/2}. \quad (1.8)$$

We show now that if $100 \leq 2^j \leq N^{1/10}$ then

$$\|(\widehat{u} * \widehat{v})(\xi, \mu, \tau) \chi_j(\tau - \omega(\xi, \mu))\|_{L_{\xi, \mu, \tau}^2} \geq c 2^{j/2} N^{-1/2}. \quad (1.9)$$

So the bound (1.8) would fail by $\ln N$ since the sum in j has $\approx \ln N$ terms. To prove (1.9), by duality, it suffices to prove that if $100 \leq 2^j \leq N^{1/10}$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} (\widehat{u} * \widehat{v})(\xi, \mu, \tau) \mathbf{1}_{[N-10, N+10]}(\xi) \mathbf{1}_{[\sqrt{3}N^2-100N, \sqrt{3}N^2+100N]}(\mu) \chi_j(\tau - \omega(\xi, \mu)) d\xi d\mu d\tau \geq c2^j, \quad (1.10)$$

where $\mathbf{1}_A$ denotes the characteristic function of the set A . We substitute the formulas (1.4) and (1.5); the left-hand side of (1.10) becomes

$$\begin{aligned} & \int_{\mathbb{R}^6} \psi(\xi_1 - N) \psi((\mu_1 - \sqrt{3}\xi_1^2)/\xi_1) \psi(\tau_1 - \omega(\xi_1, \mu_1)) \psi(\xi - \xi_1 - 4) \\ & \quad \psi((\mu - \mu_1 + \sqrt{3}(\xi - \xi_1)^2)/(\xi - \xi_1)) \psi(\tau - \tau_1 - \omega(\xi - \xi_1, \mu - \mu_1)) \\ & \quad \mathbf{1}_{[N-10, N+10]}(\xi) \mathbf{1}_{[\sqrt{3}N^2-100N, \sqrt{3}N^2+100N]}(\mu) \chi_j(\tau - \omega(\xi, \mu)) d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau. \end{aligned} \quad (1.11)$$

In this expression we make the change of variables

$$\begin{aligned} \xi_1 &= \xi_1, \mu_1 = \mu_1, \xi = \xi_1 + \xi_2, \mu = \mu_1 + \mu_2 \\ \tau_1 &= \mu_1 + \omega(\xi_1, \mu_1), \tau = \mu_2 + \mu_1 + \omega(\xi_1, \mu_1) + \omega(\xi_2, \mu_2). \end{aligned}$$

Then we notice that

$$\begin{aligned} & \psi(\xi_1 - N) \psi(\xi_2 - 4) \mathbf{1}_{[N-10, N+10]}(\xi_1 + \xi_2) = \psi(\xi_1 - N) \psi(\xi_2 - 4); \\ & \psi((\mu_1 - \sqrt{3}\xi_1^2)/\xi_1) \psi((\mu_2 + \sqrt{3}\xi_2^2)/\xi_2) \mathbf{1}_{[\sqrt{3}N^2-100N, \sqrt{3}N^2+100N]}(\mu_1 + \mu_2) \\ & = \psi((\mu_1 - \sqrt{3}\xi_1^2)/\xi_1) \psi((\mu_2 + \sqrt{3}\xi_2^2)/\xi_2) \text{ if } \xi_1 \in [N-2, N+2] \text{ and } \xi_2 \in [2, 6]. \end{aligned}$$

Thus the expression in (1.11) becomes

$$\begin{aligned} & \int_{\mathbb{R}^6} \psi(\xi_1 - N) \psi((\mu_1 - \sqrt{3}\xi_1^2)/\xi_1) \psi(\mu_1) \psi(\xi_2 - 4) \psi((\mu_2 + \sqrt{3}\xi_2^2)/\xi_2) \psi(\mu_2) \\ & \quad \chi_j(\mu_1 + \mu_2 + \Omega(\xi_1, \mu_1, \xi_2, \mu_2)) d\xi_1 d\mu_1 d\mu_1 d\xi_2 d\mu_2 d\mu_2, \end{aligned} \quad (1.12)$$

where

$$\begin{aligned} \Omega(\xi_1, \mu_1, \xi_2, \mu_2) &= \omega(\xi_1, \mu_1) + \omega(\xi_2, \mu_2) - \omega(\xi_1 + \xi_2, \mu_1 + \mu_2) \\ &= -\frac{\xi_1 \xi_2}{\xi_1 + \xi_2} \left[(\sqrt{3}\xi_1 + \sqrt{3}\xi_2)^2 - (\mu_1/\xi_1 - \mu_2/\xi_2)^2 \right]. \end{aligned} \quad (1.13)$$

We make now the nonlinear change of variables

$$\mu_1 = \sqrt{3}\xi_1^2 + \beta_1 \xi_1, \mu_2 = -\sqrt{3}\xi_2^2 + \beta_2 \xi_2, \quad (1.14)$$

with $d\mu_1 d\mu_2 = \xi_1 \xi_2 d\beta_1 d\beta_2 \approx Nd\beta_1 d\beta_2$. The expression in (1.12) is bounded from below by

$$(N/2) \int_{\mathbb{R}^6} \psi(\xi_1 - N) \psi(\beta_1) \psi(\mu_1) \psi(\xi_2 - 4) \psi(\beta_2) \psi(\mu_2) \chi_j(\mu_1 + \mu_2 + \tilde{\Omega}(\xi_1, \beta_1, \xi_2, \beta_2)) d\xi_1 d\beta_1 d\mu_1 d\xi_2 d\beta_2 d\mu_2, \quad (1.15)$$

where, by (1.13),

$$\tilde{\Omega}(\xi_1, \beta_1, \xi_2, \beta_2) = (\beta_1 - \beta_2) \xi_1 \xi_2 \left(2\sqrt{3} + \frac{\beta_1 - \beta_2}{\xi_1 + \xi_2} \right). \quad (1.16)$$

It follows from (1.16) that if $\xi_1 \in [N - 1/100, N + 1/100]$,

$$\xi_2 \in [4 - 1/100, 4 + 1/100],$$

$$|\beta_1 - \beta_2| \in [[1/(8\sqrt{3}) - 1/100]2^j/N, [1/(8\sqrt{3}) + 1/100]2^j/N],$$

$$\mu_1, \mu_2 \in [-2, 2], \text{ and } 2^j \in [100, N^{1/10}] \text{ then}$$

$$\chi_j(\mu_1 + \mu_2 + \tilde{\Omega}(\xi_1, \beta_1, \xi_2, \beta_2)) = 1.$$

Thus the only nontrivial restriction in the integral in (1.15) is

$$|\beta_1 - \beta_2| \in [[1/(8\sqrt{3}) - 1/100]2^j/N, [1/(8\sqrt{3}) + 1/100]2^j/N],$$

which shows that this integral is bounded from below by $cN \cdot 2^j/N = c2^j$. This is the bound (1.10), which implies (1.9).

1.2. The main theorem. In this section we introduce again the spaces of functions E and P already defined in [2] and state the main result that replaces Theorem 1 in [2]. We define the energy space E ,

$$E = \{\phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} : \|\phi\|_E := \|\hat{\phi}(\xi, \mu) \cdot (1 + |\xi| + |\mu/\xi|)\|_{L^2_{\xi, \mu}} < \infty\}, \quad (1.17)$$

and the weighted space P ,

$$P = \{\phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} : \|\phi\|_P := \|(y + i) \cdot \phi\|_{L^2} < \infty\}. \quad (1.18)$$

In Section 2, see (2.8), we will define a Banach space $F \hookrightarrow C(\mathbb{R} : E \cap P)$; let

$$F_1 = \{u \in C([-1, 1] : E \cap P) : \|u\|_{F_1} = \inf_{\tilde{u}=u \text{ on } \mathbb{R}^2 \times [-1, 1]} \|\tilde{u}\|_F < \infty\}.$$

For any Banach space V and $r > 0$ let $B(r, V)$ denote the open ball $\{v \in V : \|v\|_V < r\}$. Our main theorem concerns local well-posedness of the KP-I initial value problem (1.1) for small data in $E \cap P$.

Theorem 1.1. *There are $\bar{r}, \bar{R} \in (0, 1]$, $\bar{r} \leq \bar{R}$, with the property that for any $\phi \in B(\bar{r}, E \cap P)$ there is a unique $u \in B(\bar{R}, F_1)$ such that*

$$\begin{cases} (\partial_t + \partial_x^3 - \partial_x^{-1} \partial_y^2)u + \partial_x(u^2/2) = 0 \text{ in } C((-1, 1) : H^{-2}); \\ u(0) = \phi. \end{cases}$$

In addition, the mapping $\phi \rightarrow u$ is Lipschitz continuous from $B(\bar{r}, E \cap P)$ to $B(\bar{R}, F_1)$.

The rest of the paper is organized as follows: in section 2 we define the main normed spaces X_k, Y_k, V_k, W_k, F , and N , and prove some of their basic properties. As explained in subsection 1.1, the use of standard $X^{s,b}$ -type spaces seems to lead inevitably to logarithmic divergences in the modulation variable. To avoid these logarithmic divergences we work with high-frequency spaces that have two components: an $X^{s,b}$ -type component measured in the frequency space (see the space X_k) and a normalized $L_y^1 L_{x,t}^2$ component measured in the physical space (see the space Y_k). As in [3], [4], and [5], for the physical space component we use a suitable normalization of the local smoothing space $L_y^1 L_{x,t}^2$.

In section 3 we prove two linear estimates. In section 4 we prove Theorem 1.1, using a direct perturbative argument in the Banach space F_1 , and assuming the dyadic bilinear estimates (4.1) and (4.2). The remaining sections are concerned with the proofs of (4.1) and (4.2): in sections 5 and 6 we prove preliminary linear estimates and an L^2 bilinear estimate. In sections 7, 8, and 9 we prove the dyadic bilinear estimate (4.1). In section 10 we prove the dyadic bilinear estimate (4.2).

2. THE RESOLUTION SPACES

In this section we define the main normed spaces we will use in the rest of the paper, and prove some of their basic properties. Let $\mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty)$. For $k \in \mathbb{Z}$ let $I_k = \{\xi : |\xi| \in [2^{k-1}, 2^{k+1}]\}$, $\tilde{I}_k = I_k$ if $k \geq 1$, $\tilde{I}_k = [-2, 2]$ if $k = 0$, and $\tilde{I}_k = \emptyset$ if $k \leq -1$. Let $\mu_0 : \mathbb{R} \rightarrow [0, 1]$ denote an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For $k \in \mathbb{Z}$ let $\chi_k(\xi) = \mu_0(\xi/2^k) - \mu_0(\xi/2^{k-1})$. Let $\mu_k = \chi_k$ for $k \in \mathbb{Z} \cap [1, \infty)$ and $\mu_k = 0$ for $k \in \mathbb{Z} \cap (-\infty, -1]$. Let

$$\chi_{[k_1, k_2]} = \sum_{k=k_1}^{k_2} \chi_k \text{ for any } k_1 \leq k_2 \in \mathbb{Z}.$$

and, for $j \in \mathbb{Z}$,

$$\mu_{\leq j} = \sum_{j'=-\infty}^j \mu_{j'} \text{ and } \mu_{\geq j} = \sum_{j'=j}^{\infty} \mu_{j'}.$$

For $(\xi, \mu) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$ let

$$\omega(\xi, \mu) = \xi^3 + \mu^2/\xi. \quad (2.1)$$

We define the relevant KP-I region¹

$$R_{KP-I} = \{(\xi, \mu, \tau) \in \mathbb{R}^3 : |\xi| \geq 1, |\mu| \in [|\xi|^2/2^{20}, 2^{20} \cdot |\xi|^2], |\tau - \omega(\xi, \mu)| \leq |\xi|\}. \quad (2.2)$$

¹The main difficulties of the KP-I problem, including the counterexample of subsection 1.1, are caused by functions with Fourier support in this region.

For $k \in \mathbb{Z}$ let $k_+ = \max(k, 0)$. We define the normed spaces X_k ,

$$X_k = \{f \in L^2(I_k \times \mathbb{R} \times \mathbb{R}) : \|f\|_{X_k} = \sum_{j=0}^{2k_+-1} 2^{j/2} \|\mu_j(\tau - \omega(\xi, \mu)) \cdot f\|_{L^2} + \left[\sum_{j \geq 2k_+} 2^{2j-2k_+} \|\mu_j(\tau - \omega(\xi, \mu)) \cdot f\|_{L^2}^2 \right]^{1/2} < \infty\}. \quad (2.3)$$

Notice that

$$\|(\tau - \omega(\xi, \mu) + i)^{-1} \cdot \mu_{\geq J}(\tau - \omega(\xi, \mu)) \cdot f\|_{X_k} \leq C(2^{-J/2} + 2^{-2k_+/2}) \cdot \|f\|_{L^2}, \quad (2.4)$$

for any $f \in L^2(\mathbb{R}^3)$ supported in $I_k \times \mathbb{R} \times \mathbb{R}$ and $J \in \mathbb{Z}_+$.

The spaces X_k are not sufficient for a fixed-point argument, due to various logarithmic divergences. For $k \geq 100$ we also define the normed spaces Y_k ,

$$Y_k = \{f \in L^2(\mathbb{R}^3) : f \text{ supported in } R_{KP-I} \cap I_k \times \mathbb{R} \times \mathbb{R} \text{ and} \\ \|f\|_{Y_k} = 2^{-k/2} \|\mathcal{F}^{-1}[(\tau - \omega(\xi, \mu) + i) \cdot f(\xi, \mu, \tau)]\|_{L_y^1 L_{x,t}^2} < \infty\}. \quad (2.5)$$

For simplicity of notation, we define $Y_k = \{0\}$ for $k \leq 99$. Then we define the normed spaces $X_k + Y_k$, $k \in \mathbb{Z}$,

$$X_k + Y_k = \{f \in L^2(\mathbb{R}^3) : f \text{ supported in } I_k \times \mathbb{R} \times \mathbb{R} \text{ and} \\ \|f\|_{X_k + Y_k} = \inf_{f=f_1+f_2} \|f_1\|_{X_k} + \|f_2\|_{Y_k} < \infty\}.$$

For $k \in \mathbb{Z}$ we define the normed spaces V_k

$$V_k = \{f \in L^2(\mathbb{R}^3) : f \text{ supported in } I_k \times \mathbb{R} \times \mathbb{R} \text{ and} \\ \|f\|_{V_k} = \|f \cdot (1 + 2^k + i\mu/2^k)\|_{X_k + Y_k} < \infty\}, \quad (2.6)$$

and the normed spaces W_k ,

$$W_k = \{f \in L^2(\mathbb{R}^3) : f \text{ supported in } I_k \times \mathbb{R} \times \mathbb{R} \text{ and} \\ \|f\|_{W_k} = \|(\partial_\mu + I)f\|_{X_k + Y_k} < \infty\}. \quad (2.7)$$

We define the (global) normed space $F = F(\mathbb{R}^3)$,

$$F = \{u \in L^2(\mathbb{R}^3) : u \text{ supported in } \mathbb{R}^2 \times [-2, 2] \text{ and} \\ \|u\|_F^2 = \sum_{k \in \mathbb{Z}} \|\chi_k(\xi) \cdot \mathcal{F}(u)\|_{V_k \cap W_k}^2 < \infty\}, \quad (2.8)$$

and the normed space $N = N(\mathbb{R}^3)$,

$$N = \{u \in C(\mathbb{R} : H^{-2}(\mathbb{R}^3)) : \\ \|u\|_N^2 = \sum_{k \in \mathbb{Z}} \|\chi_k(\xi)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot \mathcal{F}(u)\|_{V_k \cap W_k}^2 < \infty\}. \quad (2.9)$$

We start with a simple lemma concerning basic properties of our normed spaces.

Lemma 2.1. (a) If $m : \mathbb{R} \rightarrow \mathbb{C}$, $m' : \mathbb{R}^2 \rightarrow \mathbb{C}$, $k \in \mathbb{Z}$, and f is supported in $I_k \times \mathbb{R} \times \mathbb{R}$ then

$$\begin{cases} \|m(\mu) \cdot f\|_{X_k+Y_k} \leq C \|\mathcal{F}^{-1}(m)\|_{L^1(\mathbb{R})} \cdot \|f\|_{X_k+Y_k}; \\ \|m(\mu) \cdot f\|_{V_k \cap W_k} \leq C(\|\mathcal{F}^{-1}(m)\|_{L^1(\mathbb{R})} + \|\partial_\mu m\|_{L^\infty(\mathbb{R})}) \cdot \|f\|_{V_k \cap W_k}; \\ \|m'(\xi, \tau) \cdot f\|_{X_k+Y_k} \leq C\|m'\|_{L^\infty(\mathbb{R}^2)} \|f\|_{X_k+Y_k}; \\ \|m'(\xi, \tau) \cdot f\|_{V_k \cap W_k} \leq C\|m'\|_{L^\infty(\mathbb{R}^2)} \|f\|_{V_k \cap W_k}. \end{cases} \quad (2.10)$$

(b) If $k \in \mathbb{Z}$, $j \geq 0$, and $f_k \in X_k + Y_k$ then

$$\|\mu_j(\tau - \omega(\xi, \mu)) \cdot f_k\|_{X_k} \leq C\|f_k\|_{X_k+Y_k}. \quad (2.11)$$

In particular, for any $J \in \mathbb{Z}_+$,

$$\|\eta_{\geq J}(\tau - \omega(\xi, \mu)) \cdot f_k\|_{L^2} \leq C2^{-J/2}(2^{(J-2k_+)/2} + 1)^{-1} \cdot \|f_k\|_{X_k+Y_k}, \quad (2.12)$$

and

$$\|f_k\|_{X_k} \leq C(1 + k_+)\|f_k\|_{X_k+Y_k}. \quad (2.13)$$

(c) If $k \geq 0$, $j \in [0, k] \cap \mathbb{Z}$, and f is supported in the set

$$\{(\xi, \mu, \tau) \in \mathbb{R}^3 : \xi \in I_k, |\mu| \in [2^{2k-100}, 2^{2k+100}]\},$$

then

$$\|\mathcal{F}^{-1}[\eta_{\leq j}(\tau - \omega(\xi, \mu)) \cdot f]\|_{L_y^1 L_{x,t}^2} \leq C\|\mathcal{F}^{-1}(f)\|_{L_y^1 L_{x,t}^2}. \quad (2.14)$$

Proof of Lemma 2.1. Part (a) follows directly from the definitions.

For part (b), we may assume $k \geq 100$, $f_k \in Y_k$, so f_k can be written as

$$\begin{aligned} f_k(\xi, \mu, \tau) &= 2^{k/2} \mathbf{1}_{I_k}(\xi) \chi_{[2k-30, 2k+30]}(\mu) \eta_{\leq k+1}(\tau - \omega(\xi, \mu)) \\ &\quad \times (\tau - \omega(\xi, \mu) + i)^{-1} \cdot \int_{\mathbb{R}} e^{-iy \cdot \mu} g_k(y, \xi, \tau) dy, \end{aligned} \quad (2.15)$$

with

$$\|f_k\|_{Y_k} = C\|g_k\|_{L_y^1 L_{\xi, \tau}^2}. \quad (2.16)$$

The bound (2.11) follows easily since $|\{\mu : |\tau - \omega(\xi, \mu)| \leq 2^{j+1}\}| \leq C2^{j-k}$ whenever $|\xi| \approx 2^k$, $|\mu| \approx 2^{2k}$, and $j \leq k + C$.

For part (c), using Plancherel theorem, it suffices to prove that

$$\left\| \int_{\mathbb{R}} e^{iy \cdot \mu} \chi_{[k-1, k+1]}(\xi) \chi_{[2k-110, 2k+110]}(\mu) \eta_{\leq j}(\tau - \omega(\xi, \mu)) d\mu \right\|_{L_y^1 L_{\xi, \tau}^\infty} \leq C. \quad (2.17)$$

In proving (2.17) we may assume $k \geq 100$. Then the function in the left-hand side of (2.17) is not zero only if $|\tau - \xi^3| \approx 2^{3k}$. Simple estimates using integration by parts show that

$$\left| \int_{\mathbb{R}} e^{iy \cdot \mu} \chi_{[k-1, k+1]}(\xi) \chi_{[2k-110, 2k+110]}(\mu) \eta_{\leq j}(\tau - \omega(\xi, \mu)) d\mu \right| \leq C \frac{2^{j-k}}{1 + (2^{j-k}y)^2}$$

if $|\tau - \xi^3| \approx 2^{3k}$, which suffices to prove (2.17). \square

We show now that $F \hookrightarrow C(\mathbb{R} : E \cap P)$.

Lemma 2.2. *If $u \in F$ then*

$$\sup_{t \in \mathbb{R}} \|u(\cdot, \cdot, t)\|_{E \cap P} \leq C \|u\|_F. \quad (2.18)$$

Thus $F \hookrightarrow C(\mathbb{R} : E \cap P)$.

Proof of Lemma 2.2. Let $f_k = \chi_k(\xi) \cdot \mathcal{F}(u)$, $k \in \mathbb{Z}$. In view of the definition (2.8), it suffices to prove that for any $t \in \mathbb{R}$ and $k \in \mathbb{Z}$

$$\|\mathcal{F}^{-1}(f_k)(\cdot, \cdot, t)\|_{E \cap P} \leq C \|f_k\|_{V_k \cap W_k}.$$

In view of the last bound in (2.10), we may assume $t = 0$. Thus it suffices to prove that if $k \in \mathbb{Z}$ and $f_k \in Z_k$ then

$$\left\| \int_{\mathbb{R}^3} f_k(\xi, \mu, \tau) e^{ix \cdot \xi} e^{iy \cdot \mu} d\xi d\mu d\tau \right\|_{E \cap P} \leq C \|f_k\|_{V_k \cap W_k}. \quad (2.19)$$

We show first that

$$\left\| \int_{\mathbb{R}^3} f_k(\xi, \mu, \tau) e^{ix \cdot \xi} e^{iy \cdot \mu} d\xi d\mu d\tau \right\|_E \leq C \|f_k\|_{V_k}. \quad (2.20)$$

Using the definition (1.17), it suffices to prove that

$$\left\| (1 + 2^k + i\mu/2^k) \cdot \int_{\mathbb{R}} f_k(\xi, \mu, \tau) d\tau \right\|_{L_{\xi, \mu}^2} \leq C \|f_k\|_{V_k}. \quad (2.21)$$

Using the definition (2.6), it suffices to prove that

$$\left\| \int_{\mathbb{R}} f_k(\xi, \mu, \tau) d\tau \right\|_{L_{\xi, \mu}^2} \leq C \|f_k\|_{X_k + Y_k}. \quad (2.22)$$

Assume first that $f_k \in X_k$ and write $f_k = \sum_{j \geq 0} f_k \cdot \eta_j(\tau - \omega(\xi, \mu)) = \sum_{j \geq 0} f_{k,j}$. The left-hand side of (2.22) is dominated by

$$C \sum_{j \geq 0} \left\| \int_{\mathbb{R}} |f_{k,j}(\xi, \mu, \tau)| d\tau \right\|_{L_{\xi, \mu}^2} \leq C \sum_{j \geq 0} 2^{j/2} \|f_{k,j}\|_{L_{\xi, \mu, \tau}^2} \leq C \|f_k\|_{X_k},$$

as desired. Assume now that $f_k \in Y_k$ (so $k \geq 100$) and write f_k as in (2.15). With g_k as in (2.15) and (2.16), the left-hand side of (2.22) is dominated by

$$C 2^{k/2} \left\| \mathbf{1}_{I_k}(\xi) \chi_{[2k-30, 2k+30]}(\mu) \int_{\mathbb{R} \times \mathbb{R}} e^{-iy \cdot \mu} \frac{\eta_{\leq k+1}(\tau - \omega(\xi, \mu))}{\tau - \omega(\xi, \mu) + i} \cdot g_k(y, \xi, \tau) dy d\tau \right\|_{L_{\xi, \mu}^2}. \quad (2.23)$$

We define the partial Hilbert transform operator

$$\mathcal{L}_k(g)(y, \xi, \nu) = \int_{\mathbb{R}} g(y, \xi, \tau) \cdot \eta_{\leq k+1}(\tau - \nu) \cdot (\tau - \nu + i)^{-1} d\tau.$$

Using the Minkowski inequality, the expression in (2.23) is dominated by

$$C2^{k/2} \int_{\mathbb{R}} \left\| \mathbf{1}_{I_k}(\xi) \chi_{[2k-30, 2k+30]}(\mu) \cdot \mathcal{L}_k(g_k)(y, \xi, \omega(\xi, \mu)) \right\|_{L_{\xi, \mu}^2} dy.$$

A simple change of variables shows that this is dominated by

$$C \int_{\mathbb{R}} \left\| \mathbf{1}_{I_k}(\xi) \mathcal{L}_k(g_k)(y, \xi, \nu) \right\|_{L_{\xi, \nu}^2} dy,$$

and the bound (2.22) follows from (2.16) and the estimate $\|\mathcal{L}_k(g)(y, \xi, \nu)\|_{L_{\xi, \nu}^2} \leq C\|g(y, \xi, \tau)\|_{L_{\xi, \tau}^2}$.

We show now that

$$\left\| \int_{\mathbb{R}^3} f_k(\xi, \mu, \tau) e^{ix \cdot \xi} e^{iy \cdot \mu} d\xi d\mu d\tau \right\|_P \leq C\|f_k\|_{W_k}. \quad (2.24)$$

Using the definition (1.18) and Plancherel theorem, it suffices to prove that

$$\left\| \int_{\mathbb{R}} (\partial_\mu + I) f_k(\xi, \mu, \tau) d\tau \right\|_{L_{\xi, \mu}^2} \leq C\|f_k\|_{W_k},$$

which follows from (2.22). The bound (2.19) follows from (2.20) and (2.24). \square

3. LINEAR ESTIMATES

In this section we prove two linear estimates. For $\phi \in L^2(\mathbb{R}^2)$ let $W\phi \in C(\mathbb{R} : L_{x,y}^2)$ denote the solution of the free KP-I evolution given by

$$W\phi(x, y, t) = C \int_{\mathbb{R}^2} e^{ix \cdot \xi} e^{iy \cdot \mu} e^{it\omega(\xi, \mu)} \widehat{\phi}(\xi, \mu) d\xi d\mu, \quad (3.1)$$

where $\omega(\xi, \mu)$ is defined in (2.1). Let $\psi = \widehat{\eta}_0 \in \mathcal{S}(\mathbb{R})$.

Proposition 3.1. *If $\phi \in E \cap P$ then*

$$\|\eta_0(t) \cdot W\phi\|_F \leq C\|\phi\|_{E \cap P}.$$

Proof of Proposition 3.1. A straightforward computation shows that

$$\mathcal{F}[\eta_0(t) \cdot W\phi](\xi, \mu, \tau) = \widehat{\phi}(\xi, \mu) \widehat{\eta}_0(\tau - \omega(\xi, \mu)). \quad (3.2)$$

Then, directly from the definitions,

$$\begin{aligned} \|\eta_0(t) \cdot W\phi\|_F^2 &\leq C \sum_{k \in \mathbb{Z}} \|\chi_k(\xi) \cdot \widehat{\phi}(\xi, \mu) \cdot \psi(\tau - \omega(\xi, \mu))\|_{V_k \cap W_k}^2 \\ &\leq C \sum_{k \in \mathbb{Z}} \|\chi_k(\xi) \widehat{\phi}(\xi, \mu) \cdot (1 + 2^k + |\mu|/2^k)\|_{L_{\xi, \mu}^2}^2 + C \sum_{k \in \mathbb{Z}} \|\chi_k(\xi) (\partial_\mu \widehat{\phi})(\xi, \mu)\|_{L_{\xi, \mu}^2}^2 \\ &\leq C(\|\phi\|_E^2 + \|\phi\|_P^2), \end{aligned}$$

as desired. \square

Proposition 3.2. *If $u \in N$ then*

$$\left\| \eta_0(t) \cdot \int_0^t [Wu(s)](t-s) ds \right\|_F \leq C \|u\|_N.$$

Proof of Proposition 3.2. A direct computation shows that

$$\begin{aligned} & \mathcal{F} \left[\eta_0(t) \cdot \int_0^t [Wu(s)](t-s) ds \right] (\xi, \mu, \tau) \\ &= C \int_{\mathbb{R}} \mathcal{F}(u)(\xi, \mu, \tau') \cdot \frac{\widehat{\eta}_0(\tau - \tau') - \widehat{\eta}_0(\tau - \omega(\xi, \mu))}{\tau' - \omega(\xi, \mu)} d\tau'. \end{aligned} \quad (3.3)$$

For $k \in \mathbb{Z}$ let

$$f_k(\xi, \mu, \tau') = \chi_k(\xi)(\tau' - \omega(\xi, \mu) + i)^{-1} \cdot \mathcal{F}(u)(\xi, \mu, \tau').$$

For $f_k \in V_k \cap W_k$ let

$$T(f_k)(\xi, \mu, \tau) = \int_{\mathbb{R}} f_k(\xi, \mu, \tau') \frac{\psi(\tau - \tau') - \psi(\tau - \omega(\xi, \mu))}{\tau' - \omega(\xi, \mu)} \cdot (\tau' - \omega(\xi, \mu) + i) d\tau'. \quad (3.4)$$

In view of the definitions, it suffices to prove that

$$\|T\|_{V_k \cap W_k \rightarrow V_k \cap W_k} \leq C \text{ uniformly in } k \in \mathbb{Z}. \quad (3.5)$$

We prove first that

$$\|T(f_k)\|_{X_k} \leq C \|f_k\|_{X_k} \text{ uniformly in } k \in \mathbb{Z}. \quad (3.6)$$

We observe the elementary bound

$$\left| \frac{\psi(\theta - \theta') - \psi(\theta)}{\theta'} (\theta' + i) \right| \leq C[(1 + |\theta|)^{-4} + (1 + |\theta - \theta'|)^{-4}],$$

for any $\theta, \theta' \in \mathbb{R}$. Thus, for (3.6), it suffices to prove that

$$\left\| \int_{\mathbb{R}} |f_k(\xi, \mu, \tau')| \cdot [(1 + |\tau - \tau'|)^{-4} + (1 + |\tau - \omega(\xi, \mu)|)^{-4}] d\tau' \right\|_{X_k} \leq C \|f_k\|_{X_k}, \quad (3.7)$$

for any $f \in X_k$. For this, we notice first that

$$\left\| (1 + |\tau - \omega(\xi, \mu)|)^{-4} \int_{\mathbb{R}} |f_k(\xi, \mu, \tau')| d\tau' \right\|_{X_k} \leq C \|f_k\|_{X_k},$$

using (2.22). In addition, for any $j \geq 0$,

$$\begin{aligned} & \left\| \eta_j(\tau - \omega(\xi, \mu)) \cdot \int_{\mathbb{R}} |f_k(\xi, \mu, \tau')| \cdot (1 + |\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} \\ & \leq C \sum_{j' \in \mathbb{Z}} 2^{-3|j-j'|} \|\eta_{j'}(\tau' - \omega(\xi, \mu)) \cdot f_k(\xi, \mu, \tau')\|_{L^2}. \end{aligned}$$

The bound (3.7) follows from the definition (2.3).

We show now that

$$\|T(f_k)\|_{X_k+Y_k} \leq C\|f_k\|_{Y_k} \text{ uniformly in } k \in \mathbb{Z}. \quad (3.8)$$

We may assume $k \geq 100$. Using (3.6) and Lemma 2.1 (b), (c), we may also assume that $f_k \in Y_k$ is supported in the set $\{(\xi, \mu, \tau') : |\tau' - \omega(\xi, \mu)| \leq 2^{k-10}\}$. We write

$$f_k(\xi, \mu, \tau') = \frac{\tau' - \omega(\xi, \mu)}{\tau' - \omega(\xi, \mu) + i} f_k(\xi, \mu, \tau') + \frac{i}{\tau' - \omega(\xi, \mu) + i} f_k(\xi, \mu, \tau').$$

Using Lemma 2.1 (b), $\|i(\tau' - \omega(\xi, \mu) + i)^{-1} f_k(\xi, \mu, \tau')\|_{X_k} \leq C\|f_k\|_{Y_k}$. In view of (3.6), it suffices to prove that

$$\begin{aligned} & \left\| \int_{\mathbb{R}} f_k(\xi, \mu, \tau') \psi(\tau - \tau') d\tau' \right\|_{X_k+Y_k} \\ & + \left\| \psi(\tau - \omega(\xi)) \int_{\mathbb{R}} f_k(\xi, \mu, \tau') d\tau' \right\|_{X_k} \leq C\|f_k\|_{Y_k}. \end{aligned} \quad (3.9)$$

The bound for the second term in the left-hand side of (3.9) follows from (2.22). To bound the first term we write

$$f_k(\xi, \mu, \tau') = f_k(\xi, \mu, \tau') \left[\frac{\tau' - \omega(\xi, \mu) + i}{\tau - \omega(\xi, \mu) + i} + \frac{\tau - \tau'}{\tau - \omega(\xi, \mu) + i} \right].$$

The first term in the left-hand side of (3.9) is dominated by

$$\begin{aligned} & C \left\| \eta_{\leq k-5}(\tau - \omega(\xi, \mu)) \int_{\mathbb{R}} f_k(\xi, \mu, \tau') \frac{\tau' - \omega(\xi, \mu) + i}{\tau - \omega(\xi, \mu) + i} \cdot \psi(\tau - \tau') d\tau' \right\|_{Y_k} \\ & + C \left\| \eta_{\leq k-5}(\tau - \omega(\xi, \mu)) \int_{\mathbb{R}} f_k(\xi, \mu, \tau') \frac{\psi(\tau - \tau') \cdot (\tau - \tau')}{\tau - \omega(\xi, \mu) + i} d\tau' \right\|_{X_k} \\ & + C \left\| \eta_{\geq k-5}(\tau - \omega(\xi, \mu)) \int_{\mathbb{R}} f_k(\xi, \mu, \tau') \psi(\tau - \tau') d\tau' \right\|_{X_k}. \end{aligned} \quad (3.10)$$

For the first term in (3.10), we use Lemma 2.1 (c) to bound it by

$$C2^{-k/2} \|\mathcal{F}^{-1}(\psi) \cdot \mathcal{F}^{-1}[(\tau' - \omega(\xi) + i)f_k(\xi, \mu, \tau')]\|_{L_y^1 L_{x,t}^2} \leq C\|f_k\|_{Y_k},$$

as desired. To bound the second term, we observe that

$$\left| \frac{\psi(\tau - \tau') \cdot (\tau - \tau')}{\tau - \omega(\xi, \mu) + i} \right| \leq C \frac{(1 + |\tau - \tau'|)^{-4}}{1 + |\tau' - \omega(\xi, \mu)|}.$$

Thus the second term in (3.10) is bounded by

$$C \left\| \int_{\mathbb{R}} \frac{|f_k(\xi, \mu, \tau')|}{1 + |\tau' - \omega(\xi, \mu)|} \cdot (1 + |\tau - \tau'|)^{-4} d\tau' \right\|_{X_k} \leq C \left\| \frac{f_k(\xi, \mu, \tau')}{1 + |\tau' - \omega(\xi, \mu)|} \right\|_{X_k}, \quad (3.11)$$

which is dominated by $C\|f_k\|_{Y_k}$ in view of Lemma 2.1 (b). To bound the third term in (3.10), recall that f_k is supported in the set $\{(\xi, \mu, \tau') : |\tau' - \omega(\xi, \mu)| \leq 2^{k-10}\}$.

$2^{k-10}\}$, thus $f_k(\xi, \mu, \tau') = f_k(\xi, \mu, \tau') \cdot \eta_{\leq k-10}(\tau' - \omega(\xi, \mu))$. In addition, it is easy to see that

$$|\eta_{\geq k-5}(\tau - \omega(\xi, \mu)) \cdot \eta_{\leq k-10}(\tau' - \omega(\xi, \mu)) \cdot \psi(\tau - \tau')| \leq C \frac{(1 + |\tau - \tau'|)^{-4}}{1 + |\tau' - \omega(\xi, \mu)|},$$

so the third term in (3.10) is also bounded as in (3.11).

Finally, we prove that

$$\|T(f_k)\|_{W_k} \leq C \|f_k\|_{V_k \cap W_k} \text{ uniformly in } k \in \mathbb{Z}. \quad (3.12)$$

In view of the definition (2.7), the left-hand side of (3.12) is dominated by

$$\begin{aligned} & C \|T[(\partial_\mu + I)f_k]\|_{X_k + Y_k} + C \left\| \psi'(\tau - \omega(\xi, \mu)) \cdot (\mu/\xi) \int_{\mathbb{R}} f_k(\xi, \mu, \tau') d\tau' \right\|_{X_k + Y_k} \\ & + C \left\| \int_{\mathbb{R}} f_k(\xi, \mu, \tau') \frac{d}{d\mu} \frac{\psi(\tau - \tau') - \psi(\tau - \omega(\xi, \mu))}{\tau' - \omega(\xi, \mu)} d\tau' \right\|_{X_k + Y_k}. \end{aligned}$$

The first term in the expression above is dominated by $C \|f_k\|_{W_k}$, in view of (3.6) and (3.8). The second term is dominated by $C \|f_k\|_{V_k}$, in view of (2.21). Thus, for (3.12), it suffices to prove that

$$\left\| \int_{\mathbb{R}} f_k(\xi, \mu, \tau') \cdot \frac{d}{d\mu} \frac{\psi(\tau - \tau') - \psi(\tau - \omega(\xi, \mu))}{\tau' - \omega(\xi, \mu)} d\tau' \right\|_{X_k} \leq C \|f_k\|_{V_k}. \quad (3.13)$$

By analyzing the cases $|\tau' - \omega(\xi, \mu)| \leq 1$ and $|\tau' - \omega(\xi, \mu)| \geq 1$, it is easy to see that

$$\begin{aligned} & \left| \frac{d}{d\mu} \frac{\psi(\tau - \tau') - \psi(\tau - \omega(\xi, \mu))}{\tau' - \omega(\xi, \mu)} \right| \\ & \leq \frac{C |\mu/\xi|}{1 + |\tau' - \omega(\xi, \mu)|} \cdot [(1 + |\tau - \tau'|)^{-4} + (1 + |\tau - \omega(\xi, \mu)|)^{-4}]. \end{aligned}$$

In addition, using Lemma 2.1 (b), $\|f_k \cdot (1 + |\tau - \omega(\xi, \mu)|)^{-1} \cdot |\mu/\xi|\|_{X_k} \leq C \|f_k\|_{V_k}$. Thus, for (3.13), it suffices to prove that

$$\left\| \int_{\mathbb{R}} |f_k(\xi, \mu, \tau')| \cdot [(1 + |\tau - \tau'|)^{-4} + (1 + |\tau - \omega(\xi, \mu)|)^{-4}] d\tau' \right\|_{X_k} \leq C \|f_k\|_{X_k}.$$

This follows from (3.7).

The main bound (3.5) follows from (3.6), (3.8), and (3.12). \square

The proof of Proposition 3.2 above (in particular the bounds (3.7) and (3.9)) shows that if $\varphi \in \mathcal{S}(\mathbb{R})$ then

$$\begin{cases} \|\mathcal{F}[\varphi(t) \cdot \mathcal{F}^{-1}(f)]\|_{X_k} \leq C \|f\|_{X_k}; \\ \|\mathcal{F}[\varphi(t) \cdot \mathcal{F}^{-1}(f)]\|_{X_k + Y_k} \leq C \|f\|_{X_k + Y_k}; \\ \|\mathcal{F}[\varphi(t) \cdot \mathcal{F}^{-1}(f)]\|_{V_k \cap W_k} \leq C \|f\|_{V_k \cap W_k}, \end{cases} \quad (3.14)$$

for any $k \in \mathbb{Z}$ and $f \in L^2(\mathbb{R}^3)$ supported in $I_k \times \mathbb{R} \times \mathbb{R}$.

4. PROOF OF THEOREM 1.1

In this section we reduce Theorem 1.1 to proving the following two dyadic bilinear estimates: assume $k_i \in \mathbb{Z}$, $f_{k_i} \in V_{k_i} \cap W_{k_i}$, and $\mathcal{F}(f_{k_i})$ are supported in $\mathbb{R}^2 \times [-2, 2]$, $i = 1, 2$.

- If $k \in \mathbb{Z}$, $k_1 \leq k - 20$ and $|k_2 - k| \leq 2$ then

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2}) \right\|_{V_k \cap W_k} \\ & \leq C(2^{-|k_1|/8} + 2^{-|k-k_1|/8}) \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \|f_{k_2}\|_{V_{k_2} \cap W_{k_2}}. \end{aligned} \quad (4.1)$$

- If $|k_1 - k_2| \leq 100$ then

$$\begin{aligned} & \left[\sum_{k \in \mathbb{Z}} \left\| 2^k \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2}) \right\|_{V_k \cap W_k}^2 \right]^{1/2} \\ & \leq C \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2}\|_{V_{k_2} \cap W_{k_2}}. \end{aligned} \quad (4.2)$$

Proposition 4.1. *If $u, v \in F$ then*

$$\|\partial_x(uv)\|_N \leq C \|u\|_F \cdot \|v\|_F.$$

Proof of Proposition 4.1. Let $f_k = \chi_k(\xi) \cdot \mathcal{F}(u)$ and $g_k = \chi_k(\xi) \cdot \mathcal{F}(v)$. Then

$$\begin{cases} \|u\|_F = \left[\sum_{k_1 \in \mathbb{Z}} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}}^2 \right]^{1/2}; \\ \|v\|_F = \left[\sum_{k_2 \in \mathbb{Z}} \|g_{k_2}\|_{V_{k_2} \cap W_{k_2}}^2 \right]^{1/2}. \end{cases} \quad (4.3)$$

For $k \in \mathbb{Z}$ let

$$\begin{cases} A = \{(k_1, k_2) \in \mathbb{Z}^2 : |k_1 - k_2| \leq 100\}; \\ A_1(k) = \{(k_1, k_2) \in \mathbb{Z}^2 : |k_2 - k| \leq 2 \text{ and } k_1 \leq k - 20\}; \\ A_2(k) = \{(k_1, k_2) \in \mathbb{Z}^2 : |k_1 - k| \leq 2 \text{ and } k_2 \leq k - 20\}. \end{cases}$$

Clearly

$$\chi_k(\xi) \cdot \mathcal{F}(\partial_x(uv)) = C \chi_k(\xi) \xi \sum_{(k_1, k_2) \in A \cup A_1(k) \cup A_2(k)} (f_{k_1} * g_{k_2}).$$

Thus

$$\begin{aligned} \|\partial_x(uv)\|_N^2 & \leq C \sum_{k \in \mathbb{Z}} \left(\sum_{(k_1, k_2) \in A} \left\| 2^k \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * g_{k_2}) \right\|_{V_k \cap W_k} \right)^2 \\ & + C \sum_{k \in \mathbb{Z}} \left(\sum_{(k_1, k_2) \in A_1(k)} \left\| 2^k \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * g_{k_2}) \right\|_{V_k \cap W_k} \right)^2 \\ & + C \sum_{k \in \mathbb{Z}} \left(\sum_{(k_1, k_2) \in A_2(k)} \left\| 2^k \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * g_{k_2}) \right\|_{V_k \cap W_k} \right)^2. \end{aligned} \quad (4.4)$$

Using (4.2), the first term in the right-hand side of (4.4) is bounded by

$$C \left[\sum_{(k_1, k_2) \in A} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|g_{k_2}\|_{V_{k_2} \cap W_{k_2}} \right]^2 \leq C \|u\|_F^2 \cdot \|v\|_F^2.$$

Using (4.1), the second term in the right-hand side of (4.4) is bounded by

$$C \sum_{k \in \mathbb{Z}} \left(\|u\|_F \cdot \sum_{|k_2 - k| \leq 2} \|g_{k_2}\|_{V_{k_2} \cap W_{k_2}} \right)^2 \leq C \|u\|_F^2 \cdot \|v\|_F^2.$$

The third term in the right-hand side of (4.4) is similar, and the proposition follows. \square

It follows from Proposition 3.2 and Proposition 4.1 that

$$\left\| \eta_0(t) \cdot \int_0^t [W(\partial_x(uv))(s)](t-s) ds \right\|_F \leq C \|u\|_F \cdot \|v\|_F, \quad (4.5)$$

for any $u, v \in F$. It is easy to show that F is a Banach space, and Theorem 1.1 follows from (4.5) and Proposition 3.1 by a standard fixed-point argument.

The rest of the paper is concerned with the proofs of the dyadic bilinear estimates (4.1) and (4.2).

5. PRELIMINARY ESTIMATES

In this section we prove several localized $L_y^\infty L_{x,t}^2$ and $L_y^2 L_{x,t}^\infty$ estimates and an L^4 Strichartz estimate. These bounds will be used in the bilinear estimates in Sections 7, 8, and 9. We start with a representation formula for functions in Y_k , $k \geq 100$. Let $\mathbf{1}_+$ and $\mathbf{1}_-$ denote the characteristic functions of the intervals $[0, \infty)$ and $(-\infty, 0]$ respectively.

Lemma 5.1. *If $k \geq 100$ and $f \in Y_k$, then f can be written in the form*

$$\begin{aligned} f(\xi, \mu, \tau) &= 2^{-k/2} \mathbf{1}_{I_k}(\xi) \cdot \chi_{[2k-30, 2k+30]}(M) \cdot \mathbf{1}_+(M) \\ &\quad \times \left(\frac{\eta_0(M - \mu)}{M - \mu + i/2^k} + \frac{\eta_0(M + \mu)}{M + \mu + i/2^k} \right) \cdot \int_{\mathbb{R}} e^{-iy \cdot \mu} g(y, \xi, \tau) dy + h, \end{aligned} \quad (5.1)$$

where $M = M(\xi, \tau) = \sqrt{\xi \cdot (\tau - \xi^3)}$, h is supported in the set $\{(\xi, \mu, \tau) : \xi \in I_k, |\mu| \in [2^{2k-100}, 2^{2k+100}]\}$, and

$$\|h\|_{X_k} + \|g\|_{L_y^1 L_{\xi, \tau}^2} \leq C \|f\|_{Y_k}. \quad (5.2)$$

Proof of Lemma 5.1. We start from the identity (2.15). Since $|\xi| \in [2^{k-2}, 2^{k+2}]$, $|\mu| \in [2^{2k-35}, 2^{2k+35}]$, $|\tau - \xi^3 - \mu^2/\xi| \leq 2^{k+2}$, we have $\xi \cdot (\tau - \xi^3) \in [2^{4k-80}, 2^{4k+80}]$.

So $M = M(\xi, \tau) = \sqrt{\xi \cdot (\tau - \xi^3)}$ is well-defined and $M \in [2^{2k-40}, 2^{2k+40}]$. For $\xi \in I_k$, an elementary computation shows that we can approximate

$$\begin{aligned} & \chi_{[2k-30, 2k+30]}(\mu) \cdot \mathbf{1}_+(\mu) \cdot \frac{\eta_{\leq k+1}(\tau - \omega(\xi, \mu))}{\tau - \omega(\xi, \mu) + i} \\ &= \chi_{[2k-30, 2k+30]}(M) \cdot \mathbf{1}_+(M) \cdot \frac{\xi}{2M} \cdot \frac{\eta_0(M - \mu)}{M - \mu + i/2^k} + E_+(\xi, \mu, \tau) \end{aligned}$$

where, with $\beta = |\tau - \omega(\xi, \mu)| + 1$,

$$|E_+(\xi, \mu, \tau)| \leq C \chi_{[2k-40, 2k+40]}(\mu) \cdot \frac{\eta_{\leq k+100}(\beta)}{\beta} \cdot \left(\frac{\beta}{2^k} + \frac{1}{\beta} \right). \quad (5.3)$$

Similarly, we approximate

$$\begin{aligned} & \chi_{[2k-30, 2k+30]}(\mu) \cdot \mathbf{1}_-(\mu) \cdot \frac{\eta_{\leq k+1}(\tau - \omega(\xi, \mu))}{\tau - \omega(\xi, \mu) + i} \\ &= \chi_{[2k-30, 2k+30]}(M) \cdot \mathbf{1}_-(M) \cdot \frac{\xi}{2M} \cdot \frac{\eta_0(M + \mu)}{M + \mu + i/2^k} + E_-(\xi, \mu, \tau), \end{aligned}$$

with E_- satisfying the same bound (5.3). We substitute these formulas into (2.15) and notice that the terms corresponding to E_+ and E_- can be estimated in X_k (as in the proof of Lemma 2.1 (b)). The bound (5.2) follows from (2.16). \square

We prove now a localized $L_y^\infty L_{x,t}^2$ estimate.

Lemma 5.2. *Assume $k \geq 0$, $l \geq 2k - 100$, and f is supported in the set*

$$\{(\xi, \mu, \tau) \in \mathbb{R}^3 : \xi \in I_k, |\mu| \in [2^{l-1}, 2^{l+1}]\}.$$

(a) *Then*

$$\|\mathcal{F}^{-1}(f)\|_{L_y^\infty L_{x,t}^2} \leq C 2^{-(l-k)/2} \|f\|_{X_k + Y_k}. \quad (5.4)$$

(b) *More generally, if $\varphi : \mathbb{R} \rightarrow [0, 1]$ is a smooth function supported in the interval $[-2, 2]$, $\epsilon \geq 2^{-k}$, and*

$$f_\pm^m(\xi, \mu, \tau) = f(\xi, \mu, \tau) \cdot \varphi((\mu/\xi \pm \sqrt{3}\xi)/\epsilon - m) \text{ for } m \in \mathbb{Z},$$

then

$$\left[\sum_{m \in \mathbb{Z}} \|\mathcal{F}^{-1}(f_\pm^m)\|_{L_y^\infty L_{x,t}^2}^2 \right]^{1/2} \leq C 2^{-(l-k)/2} \|f\|_{X_k + Y_k}. \quad (5.5)$$

Proof of Lemma 5.2. For part (a), assume first $f \in X_k$. Then (see [2, p. 753])

$$\|\mathcal{F}^{-1}(f)\|_{L_y^\infty L_{x,t}^2} \leq C 2^{-(l-k)/2} \|f\|_{X_k}, \quad (5.6)$$

as desired. Assume now that $f \in Y_k$, $k \geq 100$. We use the representation (5.1) and the bound (5.2). In view of (5.6), and using Plancherel's theorem, it suffices to prove that

$$\left| \int_{\mathbb{R}} e^{iy_0 \cdot \mu} \cdot \frac{\eta_0(M \pm \mu)}{M \pm \mu + i/2^k} d\mu \right| \leq C, \quad (5.7)$$

uniformly in y_0 , $M \in [2^{2k-40}, 2^{2k+40}]$, and $\xi \in I_k$. This is a standard uniform estimate for the inverse Fourier transform of a Calderón–Zygmund kernel.

For part (b), if $f \in X_k$, then (5.5) follows from (5.6) by orthogonality. Assume now $f \in Y_k$, $k \geq 100$. We use (5.1), so we may assume

$$\begin{aligned} f_{\pm}^m(\xi, \mu, \tau) &= 2^{-k/2} \mathbf{1}_{I_k}(\xi) \cdot \chi_{[2k-30, 2k+30]}(M) \cdot \mathbf{1}_+(M) \cdot \varphi((\mu/\xi \pm \sqrt{3}\xi)/\epsilon - m) \\ &\quad \times \left(\frac{\eta_0(M - \mu)}{M - \mu + i/2^k} + \frac{\eta_0(M + \mu)}{M + \mu + i/2^k} \right) \cdot \int_{\mathbb{R}} e^{-iy \cdot \mu} g(y, \xi, \tau) dy. \end{aligned}$$

By comparing the supports in μ of the functions and using the fact that $2^k \epsilon \geq 1$, we conclude that $f_{\pm}^m(\xi, \mu, \tau) \equiv 0$ unless $(\tau - \xi^3)/\xi \in [C^{-1}2^{2k}, C2^{2k}]$ and

$$\left| \frac{\sqrt{(\tau - \xi^3)/\xi} \pm \sqrt{3}\xi}{\epsilon} - m \right| \leq C_0 \text{ or } \left| \frac{-\sqrt{(\tau - \xi^3)/\xi} \pm \sqrt{3}\xi}{\epsilon} - m \right| \leq C_0.$$

We define

$$\begin{aligned} g_{\pm}^m(y, \xi, \tau) &= g(y, \xi, \tau) \\ &\quad \times \left[\eta_0 \left(\frac{\sqrt{(\tau - \xi^3)/\xi} \pm \sqrt{3}\xi}{C_0 \epsilon} - \frac{m}{C_0} \right) + \eta_0 \left(\frac{-\sqrt{(\tau - \xi^3)/\xi} \pm \sqrt{3}\xi}{C_0 \epsilon} - \frac{m}{C_0} \right) \right]. \end{aligned}$$

In view of the support property above, we have

$$\begin{aligned} f_{\pm}^m(\xi, \mu, \tau) &= 2^{-k/2} \mathbf{1}_{I_k}(\xi) \cdot \chi_{[2k-30, 2k+30]}(M) \cdot \mathbf{1}_+(M) \cdot \varphi((\mu/\xi \pm \sqrt{3}\xi)/\epsilon - m) \\ &\quad \times \left(\frac{\eta_0(M - \mu)}{M - \mu + i/2^k} + \frac{\eta_0(M + \mu)}{M + \mu + i/2^k} \right) \cdot \int_{\mathbb{R}} e^{-iy \cdot \mu} g_{\pm}^m(y, \xi, \tau) dy. \end{aligned}$$

Using part (a) (in fact a slightly modified version of the bound (5.7)),

$$\|\mathcal{F}^{-1}(f_{\pm}^m)\|_{L_y^{\infty} L_{x,t}^2} \leq C 2^{-k/2} \|g_{\pm}^m\|_{L_y^1 L_{\xi,\tau}^2}.$$

Thus, the left-hand side of (5.5) is dominated by

$$C 2^{-k/2} \left[\sum_{m \in \mathbb{Z}} \|g_{\pm}^m\|_{L_y^1 L_{\xi,\tau}^2}^2 \right]^{1/2} \leq C 2^{-k/2} \|g\|_{L_y^1 L_{\xi,\tau}^2},$$

which suffices in view of (5.2). \square

We prove now several localized maximal function estimates:

Lemma 5.3. *Assume $k, l, j \in \mathbb{Z}$, $k \leq 0$, $l \geq 0$, $j \geq 0$.*

(a) *If f is supported in the set*

$$\{(\xi, \mu, \tau) \in \mathbb{R}^3 : \xi \in I_k, |\mu| \leq 2^l, |\tau - \omega(\xi, \mu)| \leq 2^j\},$$

then

$$\|\mathcal{F}^{-1}(f)\|_{L_y^2 L_{x,t}^{\infty}} \leq C 2^{j/2} \cdot 2^{(2l+k)/4} \|(I - \partial_{\tau}^2)f\|_{L^2}. \quad (5.8)$$

(b) *If $m \in \mathbb{R}$, $\epsilon \geq 2^{-l}$, and f is supported in the set*

$$\{(\xi, \mu, \tau) \in \mathbb{R}^3 : \xi \in I_k, |\mu| \leq 2^l, |\tau - \omega(\xi, \mu)| \leq 2^j, |\mu/\xi \pm \sqrt{3}\xi - m| \leq \epsilon\},$$

then

$$\|\mathcal{F}^{-1}(f)\|_{L_y^2 L_{x,t}^\infty} \leq C 2^{j/2} \cdot (2^l \epsilon)^{1/2} \cdot 2^{k/2} \|(I - \partial_\tau^2)f\|_{L^2}. \quad (5.9)$$

Proof of Lemma 5.3. For any $f : \mathbb{R}^3 \rightarrow \mathbb{C}$ let $f^\#(\xi, \mu, \theta) = f(\xi, \mu, \theta + \omega(\xi, \mu))$. Then

$$\mathcal{F}^{-1}(f)(x, y, t) = C(t^2 + 1)^{-1} \int_{\mathbb{R}^3} [(I - \partial_\tau^2)f]^\#(\xi, \mu, \theta) e^{it\theta} e^{i(x \cdot \xi + y \cdot \mu + t \cdot \omega(\xi, \mu))} d\xi d\mu d\theta. \quad (5.10)$$

Thus, for (5.8), after noticing the time decay in (5.10), it suffices to prove that if

$$g \text{ is supported in the set } \{(\xi, \mu) : \xi \in I_k, |\mu| \leq 2^l\},$$

then

$$\left\| \int_{\mathbb{R}^2} g(\xi, \mu) e^{i(x \cdot \xi + y \cdot \mu + t \cdot \omega(\xi, \mu))} d\xi d\mu \right\|_{L_y^2 L_{x,|t| \leq 1/2}^\infty} \leq C 2^{(2l+k)/4} \|g\|_{L^2}. \quad (5.11)$$

A standard TT^* argument (see, for example, [7, p. 50]), shows that for (5.11) it suffices to prove that

$$\left\| \int_{\mathbb{R}^2} \chi_{[k-1, k+1]}^2(\xi) \eta_0^2(\mu/2^l) e^{i(x \cdot \xi + y \cdot \mu + t \cdot \omega(\xi, \mu))} d\xi d\mu \right\|_{L_y^1 L_{x,|t| \leq 1}^\infty} \leq C 2^{(2l+k)/2}. \quad (5.12)$$

To prove (5.12) we estimate the μ -integral first. Simple integration by parts and van der Corput-type arguments show that if $y \in \mathbb{R}$, $|t| \leq 1$, $|\xi| \in [2^{k-2}, 2^{k+2}]$, and k, l are as in the hypothesis then

$$\left| \int_{\mathbb{R}} \eta_0^2(\mu/2^l) e^{i(y \cdot \mu + t \cdot \mu^2/\xi)} d\mu \right| \leq C \begin{cases} 2^{l-k} |y|^{-2} & \text{if } |y| \geq 100 \cdot 2^{l-k}; \\ 2^{l/2} |y|^{-1/2} & \text{if } |y| \in [1, 100 \cdot 2^{l-k}]; \\ 2^l & \text{if } |y| \leq 1. \end{cases}$$

This leads to (5.12).

Similarly, for (5.9), it suffices to prove that if

$$g \text{ is supported in the set } \{(\xi, \mu) : \xi \in I_k, |\mu| \leq 2^l, |\mu/\xi \pm \sqrt{3}\xi - m| \leq \epsilon\},$$

then

$$\left\| \int_{\mathbb{R}^2} g(\xi, \mu) e^{i(x \cdot \xi + y \cdot \mu + t \cdot \omega(\xi, \mu))} d\xi d\mu \right\|_{L_y^2 L_{x,|t| \leq 1/2}^\infty} \leq C (2^l \epsilon)^{1/2} \cdot 2^{k/2} \|g\|_{L^2}. \quad (5.13)$$

In proving (5.13), by orthogonality, we may assume $\epsilon = 2^{-l}$. We may also assume $|m| \leq C 2^{l-k}$. As before, for (5.13), it suffices to prove that

$$\left\| \int_{\mathbb{R}^2} \chi_{[k-1, k+1]}^2(\xi) \eta_0^2(2^l(\mu/\xi \pm \sqrt{3}\xi - m)) e^{i(x \cdot \xi + y \cdot \mu + t \cdot \omega(\xi, \mu))} d\xi d\mu \right\|_{L_y^1 L_{x,|t| \leq 1}^\infty} \leq C 2^k. \quad (5.14)$$

The change of variables $\mu = \xi(\mp\sqrt{3}\xi + m + 2^{-l}\beta)$, with $d\mu = 2^{-l}\xi d\beta$, and integration by parts show that

$$\left| \int_{\mathbb{R}} \eta_0^2(2^l(\mu/\xi \pm \sqrt{3}\xi - m)) e^{i(y\mu + t\mu^2/\xi)} d\mu \right| \leq C 2^{k-l} (1 + 2^{k-l}|y|)^{-2},$$

if $y \in \mathbb{R}$, $|m| \leq C 2^{l-k}$, $|\xi| \in [2^{k-2}, 2^{k+2}]$, and $|t| \leq 1$. This leads to (5.14). \square

Lemma 5.4. *Assume $k, l, j \in \mathbb{Z}_+$.*

(a) *If f is supported in the set*

$$\{(\xi, \mu, \tau) \in \mathbb{R}^3 : \xi \in I_k, |\mu| \leq 2^l, |\tau - \omega(\xi, \mu)| \leq 2^j\},$$

then, for any $\delta > 0$,

$$\|\mathcal{F}^{-1}(f)\|_{L_y^2 L_{x,t}^\infty} \leq C_\delta 2^{j/2} \cdot (2^k + 2^{l-k})^{1/2+\delta} \|(I - \partial_\tau^2)f\|_{L^2}. \quad (5.15)$$

(b) *If $m \in \mathbb{R}$, $l \geq 2k$, $\epsilon \geq 2^{-l}$, and f is supported in the set*

$$\{(\xi, \mu, \tau) \in \mathbb{R}^3 : \xi \in I_k, |\mu| \leq 2^l, |\tau - \omega(\xi, \mu)| \leq 2^j, |\mu/\xi \pm \sqrt{3}\xi - m| \leq \epsilon\},$$

then

$$\|\mathcal{F}^{-1}(f)\|_{L_y^2 L_{x,t}^\infty} \leq C 2^{j/2} \cdot (2^l \epsilon)^{1/2} \|(I - \partial_\tau^2)f\|_{L^2}. \quad (5.16)$$

Proof of Lemma 5.3. As in the proof of Lemma 5.3, for (5.15) it suffices to show that if

$$g \text{ is supported in the set } \{(\xi, \mu) : \xi \in I_k, |\mu| \leq 2^l\},$$

then

$$\left\| \int_{\mathbb{R}^2} g(\xi, \mu) e^{i(x\cdot\xi + y\cdot\mu + t\cdot\omega(\xi, \mu))} d\xi d\mu \right\|_{L_y^2 L_{x,|t|\leq 1/2}^\infty} \leq C_\delta (2^k + 2^{l-k})^{1/2+\delta} \|g\|_{L^2}.$$

This follows from [7, Theorem 2.1 (b)].

Similarly, for (5.16), it suffices to prove that if

$$g \text{ is supported in the set } \{(\xi, \mu) : \xi \in I_k, |\mu| \leq 2^l, |\mu/\xi \pm \sqrt{3}\xi - m| \leq \epsilon\},$$

then

$$\left\| \int_{\mathbb{R}^2} g(\xi, \mu) e^{i(x\cdot\xi + y\cdot\mu + t\cdot\omega(\xi, \mu))} d\xi d\mu \right\|_{L_y^2 L_{x,|t|\leq 1/2}^\infty} \leq C (2^l \epsilon)^{1/2} \|g\|_{L^2}. \quad (5.17)$$

In proving (5.13), by orthogonality, we may assume $\epsilon = 2^{-l}$. We may also assume $|m| \leq 2^{l-k+3}$. As before, for (5.17), it suffices to prove that

$$\left\| \int_{\mathbb{R}^2} \chi_{[k-1, k+1]}^2(\xi) \eta_0^2(2^l(\mu/\xi \pm \sqrt{3}\xi - m)) e^{i(x\cdot\xi + y\cdot\mu + t\cdot\omega(\xi, \mu))} d\xi d\mu \right\|_{L_y^1 L_{x,|t|\leq 1}^\infty} \leq C. \quad (5.18)$$

We make the change of variables $\mu = \xi(\mp\sqrt{3}\xi + m + 2^{-l}\beta)$, with $d\mu = 2^{-l}\xi d\beta$. The estimate (5.18) becomes

$$2^{-l} \left\| \int_{\mathbb{R}^2} \xi \cdot \chi_{[k-1, k+1]}^2(\xi) \eta_0^2(\beta) e^{i\Phi(x, y, t, \xi, \beta)} d\xi d\beta \right\|_{L_y^1 L_{x, |t| \leq 1}^\infty} \leq C, \quad (5.19)$$

where

$$\Phi(x, y, t, \xi, \beta) = x \cdot \xi + y \cdot \xi(\mp\sqrt{3}\xi + m + 2^{-l}\beta) + t \cdot \xi^3 + t \cdot \xi(\mp\sqrt{3}\xi + m + 2^{-l}\beta)^2. \quad (5.20)$$

It remains to prove (5.19). For $|y| \leq 2^{l-k+10}$ we notice that $|\partial_\xi^3 \Phi(x, y, t, \xi, \beta)| \geq |t|$ and $|\partial_\xi^2 \Phi(x, y, t, \xi, \beta)| \geq 2\sqrt{3}|y| - C2^{l-k}|t|$, provided that $|\xi| \approx 2^k$ and $|m| \leq 2^{l-k+3}$. Thus, using van der Corput's lemma for the integral in ξ ,

$$2^{-l} \left| \int_{\mathbb{R}^2} \xi \cdot \chi_{[k-1, k+1]}^2(\xi) \eta_0^2(\beta) e^{i\Phi(x, y, t, \xi, \beta)} d\xi d\beta \right| \leq C2^{k-l} \cdot 2^{(l-k)/2} |y|^{-1/2}. \quad (5.21)$$

For $|y| \geq 2^{l-k+10}$ we integrate first by parts in β (notice that $|\partial_\beta \Phi| \geq 2^{k-l-4}|y|$ and $|\partial_\beta^2 \Phi| \leq C2^{k-2l}$ if $|t| \leq 1$). Then we use van der Corput's lemma for the integral in ξ as before. The result is

$$2^{-l} \left| \int_{\mathbb{R}^2} \xi \cdot \chi_{[k-1, k+1]}^2(\xi) \eta_0^2(\beta) e^{i\Phi(x, y, t, \xi, \beta)} d\xi d\beta \right| \leq C2^{k-l} \cdot (2^{k-l}|y|)^{-1} \cdot |y|^{-1/2}. \quad (5.22)$$

The bound (5.19) follows from (5.21) and (5.22). \square

We conclude this section with an L^4 estimate.

Lemma 5.5. *If $k \in \mathbb{Z}$ and $f \in X_k + Y_k$ then*

$$\|\mathcal{F}^{-1}(f)\|_{L_{x,y,t}^4} \leq C\|f\|_{X_k + Y_k}. \quad (5.23)$$

Proof of Lemma 5.5. We use the scale-invariant Strichartz estimate of [1]:

$$\left\| \int_{\mathbb{R}^2} \phi(\xi, \mu) e^{ix \cdot \xi} e^{iy \cdot \mu} e^{it \cdot \omega(\xi, \mu)} d\xi d\mu \right\|_{L_{x,y,t}^4} \leq C\|\phi\|_{L^2}, \quad (5.24)$$

for any $\phi \in L^2(\mathbb{R}^2)$.

Assume first that $f \in X_k$. With $f^\#$ defined as in the proof of Lemma 5.3, for $j \geq 0$

$$\begin{aligned} & \left\| \int_{\mathbb{R}^3} f(\xi, \mu, \tau) \cdot \eta_j(\tau - \omega(\xi, \mu)) \cdot e^{ix \cdot \xi} e^{iy \cdot \mu} e^{it \cdot \tau} d\xi d\mu d\tau \right\|_{L_{x,y,t}^4} \\ &= \left\| \int_{\mathbb{R}^3} f^\#(\xi, \mu, \theta) \cdot \eta_j(\theta) e^{it \cdot \theta} \cdot e^{ix \cdot \xi} e^{iy \cdot \mu} e^{it \cdot \omega(\xi, \mu)} d\xi d\mu d\theta \right\|_{L_{x,y,t}^4} \\ &\leq C2^{j/2} \|f^\#(\xi, \mu, \theta) \cdot \eta_j(\theta)\|_{L^2}, \end{aligned} \quad (5.25)$$

which gives (5.23).

Assume now that $f \in Y_k$. We use the representation (5.1). With the notation in Lemma 5.1, using (5.2) and the bound (5.23) for $f \in X_k$, it suffices to prove that

$$2^{-k/2} \left\| \int_{\mathbb{R}^3} g(\xi, \tau) \mathbf{1}_{I_k}(\xi) \cdot \chi_{[2k-30, 2k+30]}(M) \cdot \mathbf{1}_+(M) \right. \\ \left. \times \frac{\eta_0(M \pm \mu)}{M \pm \mu + i/2^k} \cdot e^{ix \cdot \xi} e^{iy \cdot \mu} e^{it \cdot \tau} d\xi d\mu d\tau \right\|_{L^4_{x,y,t}} \leq C \|g\|_{L^2},$$

for any $g \in L^2(\mathbb{R}^2)$. We take the integral in μ first; it remains to prove that

$$\left\| \int_{\mathbb{R}^2} g(\xi, \tau) \mathbf{1}_{I_k}(\xi) \cdot \chi_{[2k-30, 2k+30]}(M) \cdot \mathbf{1}_+(M) \right. \\ \left. \times e^{ix \cdot \xi} e^{iy \cdot M} e^{it \cdot \tau} d\xi d\tau \right\|_{L^4_{x,y,t}} \leq C 2^{k/2} \|g\|_{L^2}.$$

We make the change of variables $\tau = \xi^3 + \nu^2/\xi$, $\nu \in [C^{-1}2^{2k}, C2^{2k}]$, $d\tau = 2(\nu/\xi)d\nu$. Clearly, $M(\xi, \tau) = \nu$. Thus, it suffices to prove that

$$\left\| \int_{\mathbb{R}^2} g(\xi, \xi^3 + \nu^2/\xi) \mathbf{1}_{I_k}(\xi) \cdot \chi_{[2k-30, 2k+30]}(\nu) \right. \\ \left. \times \mathbf{1}_+(\nu) e^{ix \cdot \xi} e^{iy \cdot \nu} e^{it \cdot \omega(\xi, \nu)} d\xi d\nu \right\|_{L^4_{x,y,t}} \leq C 2^{-k/2} \|g\|_{L^2}.$$

This follows from (5.24) with $\phi(\xi, \nu) = g(\xi, \xi^3 + \nu^2/\xi) \mathbf{1}_{I_k}(\xi) \cdot \chi_{[2k-30, 2k+30]}(\nu) \mathbf{1}_+(\nu)$. \square

6. AN L^2 BILINEAR ESTIMATE

In this section we prove an L^2 bilinear estimate. For $k \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$ let

$$D_{k,j} = \{(\xi, \mu, \tau) : \xi \in I_k, \mu \in \mathbb{R}, |\tau - \omega(\xi, \mu)| \leq 2^j\}.$$

Lemma 6.1. *Assume $k_1, k_2, k_3 \in \mathbb{Z}$, $j_1, j_2, j_3 \in \mathbb{Z}_+$, and $f_i : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ are L^2 functions supported in D_{k_i, j_i} , $i = 1, 2, 3$. If*

$$\max(j_1, j_2, j_3) \leq k_1 + k_2 + k_3 - 20 \quad (6.1)$$

then

$$\int_{\mathbb{R}^3} (f_1 * f_2) \cdot f_3 \leq C 2^{(j_1 + j_2 + j_3)/2} \cdot 2^{-(k_1 + k_2 + k_3)/2} \cdot \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}. \quad (6.2)$$

Before we proceed to the proof of this lemma we state a simple corollary that follows by duality.

Corollary 6.2. *Assume $k_1, k_2, k_3 \in \mathbb{Z}$, $j_1, j_2, j_3 \in \mathbb{Z}_+$, and $f_i : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ are L^2 functions supported in D_{k_i, j_i} , $i = 1, 2$. If*

$$\max(j_1, j_2, j_3) \leq k_1 + k_2 + k_3 - 20$$

then

$$\|(f_1 * f_2) \cdot \mathbf{1}_{D_{k_3, j_3}}\|_{L^2} \leq C 2^{(j_1+j_2+j_3)/2} \cdot 2^{-(k_1+k_2+k_3)/2} \cdot \|f_1\|_{L^2} \|f_2\|_{L^2}.$$

Proof of Lemma 6.1. Clearly,

$$\int_{\mathbb{R}^3} (f_1 * f_2) \cdot f_3 = \int_{\mathbb{R}^3} (\tilde{f}_1 * f_3) \cdot f_2 = \int_{\mathbb{R}^3} (\tilde{f}_2 * f_3) \cdot f_1,$$

where $\tilde{f}_i(\xi, \mu, \tau) = f_i(-\xi, -\mu, -\tau)$, $i = 1, 2$. In view of the symmetry of (6.2) we may assume

$$j_3 = \max(j_1, j_2, j_3). \quad (6.3)$$

As in the proof of Lemma 5.3, we define $f_i^\#(\xi, \mu, \theta) = f_i(\xi, \mu, \theta + \omega(\xi, \mu))$, $i = 1, 2, 3$, $\|f_i^\#\|_{L^2} = \|f_i\|_{L^2}$. We rewrite the left-hand side of (6.2) in the form

$$\begin{aligned} & \int_{\mathbb{R}^6} f_1^\#(\xi_1, \mu_1, \theta_1) \cdot f_2^\#(\xi_2, \mu_2, \theta_2) \\ & \times f_3^\#(\xi_1 + \xi_2, \mu_1 + \mu_2, \theta_1 + \theta_2 + \Omega((\xi_1, \mu_1), (\xi_2, \mu_2))) d\xi_1 d\xi_2 d\mu_1 d\mu_2 d\theta_1 d\theta_2, \end{aligned} \quad (6.4)$$

where

$$\begin{aligned} \Omega((\xi_1, \mu_1), (\xi_2, \mu_2)) &= -\omega(\xi_1 + \xi_2, \mu_1 + \mu_2) + \omega(\xi_1, \mu_1) + \omega(\xi_2, \mu_2) \\ &= \frac{-\xi_1 \xi_2}{\xi_1 + \xi_2} \left[(\sqrt{3}\xi_1 + \sqrt{3}\xi_2)^2 - \left(\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right)^2 \right]. \end{aligned} \quad (6.5)$$

The functions $f_i^\#$ are supported in the sets $\{\xi, \mu, \theta) : \xi \in I_{k_i}, \mu \in \mathbb{R}, |\theta| \leq 2^{j_i}\}$.

We will prove that if $g_i : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ are L^2 functions supported in $I_{k_i} \times \mathbb{R}$, $i = 1, 2$, and $g : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ is an L^2 function supported in $I_k \times \mathbb{R} \times [-2^j, 2^j]$, $j \leq k_1 + k_2 + k - 15$, then

$$\begin{aligned} & \int_{\mathbb{R}^4} g_1(\xi_1, \mu_1) \cdot g_2(\xi_2, \mu_2) \cdot g(\xi_1 + \xi_2, \mu_1 + \mu_2, \Omega((\xi_1, \mu_1), (\xi_2, \mu_2))) d\xi_1 d\xi_2 d\mu_1 d\mu_2 \\ & \leq C 2^{j/2} \cdot 2^{-(k_1+k_2+k)/2} \cdot \|g_1\|_{L^2} \|g_2\|_{L^2} \|g\|_{L^2}. \end{aligned} \quad (6.6)$$

This suffices for (6.2), in view of (6.3) and (6.4).

To prove (6.6), we observe² first that we may assume that the integral in the left-hand side of (6.6) is taken over the set

$$\mathcal{R}_{++} = \{(\xi_1, \mu_1, \xi_2, \mu_2) : \xi_1 + \xi_2 \geq 0 \text{ and } \mu_1/\xi_1 - \mu_2/\xi_2 \geq 0\}.$$

Using the restriction $j \leq k_1 + k_2 + k - 15$ and (6.5), we may assume also that the integral in the left-hand side of (6.6) is taken over the set

$$\tilde{\mathcal{R}}_{++} = \{(\xi_1, \mu_1, \xi_2, \mu_2) \in \mathcal{R}_{++} : |\sqrt{3}(\xi_1 + \xi_2)| - |\mu_1/\xi_1 - \mu_2/\xi_2| \leq 2^{-10}|\xi_1 + \xi_2|\}.$$

²There are four identical integrals of this type.

To summarize, it suffices to prove that

$$\begin{aligned} \int_{\tilde{\mathcal{R}}_{++}} g_1(\xi_1, \mu_1) \cdot g_2(\xi_2, \mu_2) \cdot g(\xi_1 + \xi_2, \mu_1 + \mu_2, \Omega((\xi_1, \mu_1), (\xi_2, \mu_2))) d\xi_1 d\xi_2 d\mu_1 d\mu_2 \\ \leq C 2^{j/2} \cdot 2^{-(k_1+k_2+k)/2} \cdot \|g_1\|_{L^2} \|g_2\|_{L^2} \|g\|_{L^2}. \end{aligned} \quad (6.7)$$

We make the changes of variables

$$\mu_1 = \sqrt{3}\xi_1^2 + \beta_1\xi_1 \text{ and } \mu_2 = -\sqrt{3}\xi_2^2 + \beta_2\xi_2,$$

with $d\mu_1 d\mu_2 = \xi_1 \xi_2 d\beta_1 d\beta_2$. The left-hand side of (6.7) is bounded by

$$\begin{aligned} C 2^{k_1+k_2} \int_S g_1(\xi_1, \sqrt{3}\xi_1^2 + \beta_1\xi_1) \cdot g_2(\xi_2, -\sqrt{3}\xi_2^2 + \beta_2\xi_2) \\ \times g(\xi_1 + \xi_2, \sqrt{3}\xi_1^2 - \sqrt{3}\xi_2^2 + \beta_1\xi_1 + \beta_2\xi_2, \tilde{\Omega}((\xi_1, \beta_1), (\xi_2, \beta_2))) d\xi_1 d\xi_2 d\beta_1 d\beta_2, \end{aligned} \quad (6.8)$$

where

$$S = \{(\xi_1, \beta_1, \xi_2, \beta_2) : \xi_1 + \xi_2 \geq 0 \text{ and } |\beta_1 - \beta_2| \leq 2^{-10}(\xi_1 + \xi_2)\}, \quad (6.9)$$

and

$$\tilde{\Omega}((\xi_1, \beta_1), (\xi_2, \beta_2)) = \xi_1 \xi_2 (\beta_1 - \beta_2) \left(2\sqrt{3} + \frac{\beta_1 - \beta_2}{\xi_1 + \xi_2} \right). \quad (6.10)$$

We define the functions $h_i : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ supported in $I_{k_i} \times \mathbb{R}$, $i = 1, 2$,

$$\begin{cases} h_1(\xi_1, \beta_1) = 2^{k_1/2} \cdot g_1(\xi_1, \sqrt{3}\xi_1^2 + \beta_1\xi_1); \\ h_2(\xi_2, \beta_2) = 2^{k_2/2} \cdot g_2(\xi_2, -\sqrt{3}\xi_2^2 + \beta_2\xi_2), \end{cases}$$

with $\|h_i\|_{L^2} \approx \|g_i\|_{L^2}$. Thus, for (6.6) it suffices to prove that

$$\begin{aligned} 2^{(k_1+k_2)/2} \int_S h_1(\xi_1, \beta_1) \cdot h_2(\xi_2, \beta_2) \\ \times g(\xi_1 + \xi_2, \sqrt{3}\xi_1^2 - \sqrt{3}\xi_2^2 + \beta_1\xi_1 + \beta_2\xi_2, \tilde{\Omega}((\xi_1, \beta_1), (\xi_2, \beta_2))) d\xi_1 d\xi_2 d\beta_1 d\beta_2 \\ \leq C 2^{j/2} \cdot 2^{-(k_1+k_2+k)/2} \cdot \|h_1\|_{L^2} \|h_2\|_{L^2} \|g\|_{L^2}. \end{aligned} \quad (6.11)$$

To prove (6.11), we may assume without loss of generality that

$$k_1 \leq k_2. \quad (6.12)$$

We make the change of variables $\beta_1 = \beta_2 + \beta$. In view of (6.9), (6.10), and the restriction on the support of g , we may assume $|\beta| \leq 2^{j-k_1-k_2+4}$. Thus, the integral in the left-hand side of (6.11) is equal to

$$\begin{aligned} 2^{(k_1+k_2)/2} \int_{\tilde{S}} h_1(\xi_1, \beta + \beta_2) \cdot h_2(\xi_2, \beta_2) \cdot \mathbf{1}_{[-1,1]}(\beta/2^{j-k_1-k_2+4}) \\ \times g(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta)) d\xi_1 d\xi_2 d\beta d\beta_2, \end{aligned} \quad (6.13)$$

where $\tilde{S} = \{(\xi_1, \xi_2, \beta, \beta_2) \in \mathbb{R}^4 : \xi_1 + \xi_2 \geq 0 \text{ and } |\beta| \leq 2^{-10}(\xi_1 + \xi_2)\}$, and

$$\begin{cases} A(\xi_1, \xi_2, \beta) = \sqrt{3}\xi_1^2 - \sqrt{3}\xi_2^2 + \beta\xi_1; \\ B(\xi_1, \xi_2, \beta) = \xi_1\xi_2\beta \cdot (2\sqrt{3} + \beta/(\xi_1 + \xi_2)). \end{cases} \quad (6.14)$$

Let $j' = j - k_1 - k_2 + 4$ and decompose, for $i = 1, 2$,

$$h_i(\xi', \beta') = \sum_{m \in \mathbb{Z}} h_i(\xi', \beta') \cdot \mathbf{1}_{[0,1)}(\beta'/2^{j'} - m) = \sum_{m \in \mathbb{Z}} h_i^m(\xi', \beta').$$

The expression in (6.13) is dominated by to

$$\begin{aligned} & 2^{(k_1+k_2)/2} \sum_{|m-m'|\leq 4} \int_{\tilde{S}} h_1^m(\xi_1, \beta + \beta_2) \cdot h_2^{m'}(\xi_2, \beta_2) \\ & \times g(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta)) d\xi_1 d\xi_2 d\beta d\beta_2. \end{aligned} \quad (6.15)$$

Also, for $i = 1, 2$,

$$\|h_i\|_{L^2} = \left[\sum_{m \in \mathbb{Z}} \|h_i^m\|_{L^2}^2 \right].$$

Thus, to prove (6.11), we may assume $h_1 = h_1^m$ and $h_2 = h_2^{m'}$ for some fixed $m, m' \in \mathbb{Z}$ with $|m - m'| \leq 4$. To summarize, it suffices to prove that if $F_i : \mathbb{R}^2 \rightarrow [0, \infty)$ are L^2 functions supported in $I_{k_i} \times \mathbb{R}$, g is as before, and $m \in \mathbb{Z}$ then

$$\begin{aligned} & 2^{(k_1+k_2)/2} \int_{\tilde{S}} F_1(\xi_1, \beta + \beta_2) \cdot F_2(\xi_2, \beta_2) \cdot \mathbf{1}_{[m-1, m+1]}(\beta_2/2^{j'}) \\ & \times g(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta)) d\xi_1 d\xi_2 d\beta d\beta_2 \\ & \leq C 2^{j/2} \cdot 2^{-(k_1+k_2+k)/2} \cdot \|F_1\|_{L^2} \|F_2\|_{L^2} \|g\|_{L^2}. \end{aligned} \quad (6.16)$$

To prove (6.16) we use the Cauchy-Schwartz inequality in the variables (ξ_1, ξ_2, β) : with

$$S' = \{(\xi_1, \xi_2, \beta) \in \mathbb{R}^3 : \xi_i \in I_{k_i}, \xi_1 + \xi_2 \geq 0, |\beta| \leq 2^{-10}(\xi_1 + \xi_2)\},$$

the left-hand side of (6.16) is dominated by

$$\begin{aligned} & C 2^{(k_1+k_2)/2} \int_{\mathbb{R}} \mathbf{1}_{[m-1, m+1]}(\beta_2/2^{j'}) \cdot \left(\int_{S'} |F_1(\xi_1, \beta + \beta_2) \cdot F_2(\xi_2, \beta_2)|^2 d\xi_1 d\xi_2 d\beta \right)^{1/2} \\ & \times \left(\int_{S'} |g(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta))|^2 d\xi_1 d\xi_2 d\beta \right)^{1/2} d\beta_2. \end{aligned} \quad (6.17)$$

For (6.16), it is easy to see that it suffices to prove that

$$\begin{aligned} & \left(\int_{S'} |g(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta))|^2 d\xi_1 d\xi_2 d\beta \right)^{1/2} \\ & \leq C 2^{-(k_1+k_2+k)/2} \|g\|_{L^2}. \end{aligned} \quad (6.18)$$

for any $\beta_2 \in \mathbb{R}$. Indeed, assuming (6.18), we can bound the expression in (6.17) by

$$C2^{(k_1+k_2)/2} \int_{\mathbb{R}} \mathbf{1}_{[m-1, m+1]}(\beta_2/2^{j'}) \cdot \|F_1\|_{L^2} \|F_2(\cdot, \beta_2)\|_{L_{\xi_2}^2} \cdot 2^{-(k_1+k_2+k)/2} \|g\|_{L^2} d\beta_2,$$

which suffices since $2^{j'/2} 2^{(k_1+k_2)/2} \approx 2^{j/2}$.

Finally, to prove (6.18), we may assume first that $\beta_2 = 0$. We examine (6.14) and make the change of variable $\beta = \sqrt{3}(\xi_1 + \xi_2) \cdot \nu$. The left-hand side of (6.18) is dominated by

$$C \left(2^k \int_{S''} |g(\xi_1 + \xi_2, \sqrt{3}(\xi_1 + \xi_2)(\xi_1 - \xi_2 + \nu\xi_1), 3\xi_1\xi_2(\xi_1 + \xi_2)\nu(2 + \nu))|^2 d\xi_1 d\xi_2 d\nu \right)^{1/2}, \quad (6.19)$$

where $S'' = \{(\xi_1, \xi_2, \nu) \in \mathbb{R}^3 : \xi_i \in I_{k_i}, |\nu| \leq 2^{-10}\}$. We define the function

$$h(\xi, x, y) = 2^{2k} \cdot |g(\xi, \sqrt{3}\xi \cdot x, 3\xi \cdot y)|^2,$$

so $\|h\|_{L^1} \approx \|g\|_{L^2}^2$. The expression in (6.19) is dominated by

$$C2^{-k/2} \left(\int_{S''} |h(\xi_1 + \xi_2, \xi_1 - \xi_2 + \nu\xi_1, \xi_1\xi_2 \cdot \nu(2 + \nu))| d\xi_1 d\xi_2 d\nu \right)^{1/2}.$$

Therefore, it remains to prove that

$$\int_{S''} |h(\xi_1 + \xi_2, \xi_1 - \xi_2 + \nu\xi_1, \xi_1\xi_2 \cdot \nu(2 + \nu))| d\xi_1 d\xi_2 d\nu \leq C2^{-(k_1+k_2)} \|h\|_{L^1}$$

for any function $h \in L^1(\mathbb{R}^3)$. This is clear since the absolute value of the determinant of the change of variables $(\xi_1, \xi_2, \nu) \rightarrow [\xi_1 + \xi_2, \xi_1 - \xi_2 + \nu\xi_1, \xi_1\xi_2 \cdot \nu(2 + \nu)]$ is equal to $(2 + \nu)|\xi_1| \cdot |\xi_2(2 + \nu) + \xi_1\nu| \approx 2^{k_1+k_2}$, see (6.12) and the definition of the set S'' . \square

7. DYADIC BILINEAR ESTIMATES I

In this section we prove the bound (4.1) for $k \geq 40$ and $k_1 \in [0, k - 20]$.

Proposition 7.1. *Assume $k \geq 40$, $k_2 \in [k - 2, k + 2]$, $k_1 \in [0, k - 20]$, $f_{k_1} \in V_{k_1} \cap W_{k_1}$, $f_{k_2} \in V_{k_2} \cap W_{k_2}$, and $\mathcal{F}^{-1}(f_{k_1})(x, y, t)$ is supported in $\mathbb{R}^2 \times [-2, 2]$. Then*

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2}) \right\|_{V_k \cap W_k} \\ & \leq C(2^{-k_1/8} + 2^{-(k-k_1)/8}) \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \|f_{k_2}\|_{V_{k_2} \cap W_{k_2}}. \end{aligned} \quad (7.1)$$

Proposition 7.1 follows from Lemma 7.2, Lemma 7.3, and Lemma 7.4 below. We start by decomposing³

$$f_{k_2} = f_{k_2, 2k_2-10} + \sum_{l_2 \geq 2k_2-9} f_{k_2, l_2} = f_{k_2} \cdot \eta_{\leq 2k_2-10}(\mu_2) + \sum_{l_2 \geq 2k_2-9} f_{k_2} \cdot \eta_{l_2}(\mu_2).$$

and

$$f_{k_1} = f_{k_1, 2k_1} + \sum_{l_1 \geq 2k_1+1} f_{k_1, l_1} = f_{k_1} \cdot \eta_{\leq 2k_1}(\mu_1) + \sum_{l_1 \geq 2k_1+1} f_{k_1} \cdot \eta_{l_1}(\mu_1).$$

Finally for any $J \in \mathbb{Z}$ let $f_{k_i, l_i, J} = f_{k_i, l_i} \cdot \eta_J(\tau - \omega(\xi, \mu))$, $f_{k_i, l_i, \leq J} = f_{k_i, l_i} \cdot \eta_{\leq J}(\tau - \omega(\xi, \mu))$, and $f_{k_i, l_i, > J} = f_{k_i, l_i} \cdot \eta_{\geq J+1}(\tau - \omega(\xi, \mu))$, $i = 1, 2$.

Lemma 7.2. *With the notation in Proposition 7.1, for any $l_2 \in [2k_2 - 9, 2k_2 + 9]$*

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, l_2}) \right\|_{V_k \cap W_k} \\ & \leq C(2^{-k_1/8} + 2^{-(k-k_1)/8}) \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \|f_{k_2, l_2}\|_{V_{k_2} \cap W_{k_2}}. \end{aligned}$$

Proof of Lemma 7.2. In view of the definitions and Lemma 2.1 (b), it suffices to prove that

$$\begin{aligned} & \left\| \chi_k(\xi) \cdot (2^{2k} + i\mu)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, l_2}) \right\|_{X_k + Y_k} \\ & + 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * (\partial_\mu + I)f_{k_2, l_2}) \right\|_{X_k + Y_k} \\ & \leq C(2^{-k_1/8} + 2^{-(k-k_1)/8}) \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} (2^k \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}} + \|(\partial_\mu + I)f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}). \end{aligned}$$

For this, it suffices to prove that

$$\begin{aligned} & \left\| \chi_k(\xi) \cdot (2^k + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, l_2}) \right\|_{X_k + Y_k} \\ & \leq C(2^{-k_1/8} + 2^{-(k-k_1)/8}) \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned} \tag{7.2}$$

In view of Lemma 2.1 (a) and (b), Lemma 5.4 (a), (3.14), and the support assumption on $\mathcal{F}^{-1}(f_{k_1})$,

$$\begin{aligned} \|\mathcal{F}^{-1}(f_{k_1, l_1, > J})\|_{L_y^2 L_{x,t}^\infty} & \leq C \sum_{j > J} 2^{j/2} 2^{(l_1 - k_1) \cdot 3/5} \|(I - \partial_{\tau_1}^2) f_{k_1, l_1, j}\|_{L^2} \\ & \leq C 2^{(l_1 - k_1) \cdot 3/5} (1 + 2^{(J - 2k_1)/2})^{-1} \|(I - \partial_{\tau_1}^2) f_{k_1, l_1}\|_{X_{k_1}} \\ & \leq C(k_1 + 1) 2^{-(l_1 - k_1) \cdot 2/5} (1 + 2^{(J - 2k_1)/2})^{-1} \cdot \|f_{k_1}\|_{V_{k_1}} \end{aligned} \tag{7.3}$$

³In the decomposition below we make an abuse of notation when we write that $f_{k_i, 2k_i} = \sum_{l_i < 2k_i+1} f_{k_i, l_i}$. One can see in the rest of the paper that this notation avoids some unnecessary technicalities. One example of its efficiency is in the fact that for any $l_i < 2k_i + 1$

$$(1 + |\xi| + |\mu/\xi|) |f_{k_i, l_i}| \sim (1 + 2^{k_i}) |f_{k_i, l_i}|$$

and hence we can simply write

$$(1 + |\xi| + |\mu/\xi|) |f_{k_i, 2k_i}| \sim (1 + 2^{k_i}) |f_{k_i, 2k_i}|.$$

Our notation also explains why in the proof of the lemmas below we will always assume that $l_1 \geq 2k_1$.

for any $l_1 \geq 2k_1$ and $J \in \mathbb{Z} \cap [-1, \infty)$.

We estimate first the contribution of $f_{k_1, l_1} * f_{k_2, l_2}$, $2k_1 \leq l_1 \leq k + k_1 - 10$. In this range we will show that

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k + Y_k} \\ & \leq C 2^{-(l_1 - k_1)/8} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned} \quad (7.4)$$

Let

$$J_0 \text{ denote the smallest integer } \geq k - (l_1 - k_1)/2 - 10. \quad (7.5)$$

Using (2.4), Lemma 2.1 (a), Lemma 5.2 (a), and (7.3) with $J = -1$, we estimate

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\geq J_0+1} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k} \\ & \leq C 2^k \cdot 2^{-J_0/2} \|f_{k_1, l_1} * f_{k_2, l_2}\|_{L_{\xi, \mu, \tau}^2} \\ & \leq C 2^k 2^{-J_0/2} \|\mathcal{F}^{-1}(f_{k_1, l_1})\|_{L_y^2 L_{x, t}^\infty} \cdot \|\mathcal{F}^{-1}(f_{k_2, l_2})\|_{L_y^\infty L_{x, t}^2} \\ & \leq C 2^{-(l_1 - k_1)/8} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned} \quad (7.6)$$

We decompose

$$\begin{aligned} f_{k_2, l_2} &= f_{k_2, l_2, \leq J_0}^+ + f_{k_2, l_2, \leq J_0}^- + f_{k_2, l_2, > J_0} = f_{k_2, l_2} \cdot \eta_{\leq J_0} (\tau_2 - \omega(\xi_2, \mu_2)) \mathbf{1}_+(\mu_2) \\ & \quad + f_{k_2, l_2} \cdot \eta_{\leq J_0} (\tau_2 - \omega(\xi_2, \mu_2)) \mathbf{1}_-(\mu_2) \\ & \quad + f_{k_2, l_2} \cdot \eta_{\geq J_0+1} (\tau_2 - \omega(\xi_2, \mu_2)) \end{aligned} \quad (7.7)$$

Using (2.12),

$$\|f_{k_2, l_2, > J_0}\|_{L^2} \leq C 2^{-J_0/2} \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \quad (7.8)$$

Thus, using the definitions, Lemma 2.1 (a), (c), and (7.3) we estimate

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2, > J_0}) \right\|_{Y_k} \\ & \leq C 2^{k/2} \cdot \|\mathcal{F}^{-1}(f_{k_1, l_1} * f_{k_2, l_2, > J_0})\|_{L_y^1 L_{x, t}^2} \\ & \leq C 2^{k/2} \|\mathcal{F}^{-1}(f_{k_1, l_1})\|_{L_y^2 L_{x, t}^\infty} \cdot \|\mathcal{F}^{-1}(f_{k_2, l_2, > J_0})\|_{L_y^2 L_{x, t}^2} \\ & \leq C 2^{-(l_1 - k_1)/8} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned} \quad (7.9)$$

An estimate similar to (7.9), using (7.3) gives

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, > k+2k_1-10} * f_{k_2, l_2, \leq J_0}^\pm) \right\|_{Y_k} \\ & \leq C 2^{-(l_1 - k_1)/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned} \quad (7.10)$$

It remains to estimate

$$2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq k+2k_1-10} * f_{k_2, l_2, \leq J_0}^\pm) \right\|_{X_k + Y_k}.$$

For $j_2 \in \mathbb{Z}_+$ let $f_{k_2, l_2, j_2}^\pm = f_{k_2, l_2} \cdot \eta_{j_2}(\tau_2 - \omega(\xi_2, \mu_2)) \cdot \mathbf{1}_\pm(\mu_2)$. Using Corollary 6.2, Lemma 2.1 (b), and the definitions, we estimate

$$\begin{aligned}
& 2^k \sum_{j_1=J_0+1}^{k+2k_1-10} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, j_1} * f_{k_2, l_2, \leq J_0}^\pm) \right\|_{X_k} \\
& \leq C 2^k \sum_{j_1=J_0+1}^{k+2k_1-10} \sum_{j, j_2=0}^{J_0} 2^{-j/2} \left\| \eta_j(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, j_1} * f_{k_2, l_2, j_2}^\pm) \right\|_{L^2} \\
& \leq C 2^k \sum_{j_1=J_0+1}^{k+2k_1-10} \sum_{j, j_2=0}^{J_0} 2^{-(2k+k_1)/2} \cdot 2^{j_1/2} \|f_{k_1, l_1, j_1}\|_{L^2} \cdot 2^{j_2/2} \|f_{k_2, l_2, j_2}^\pm\|_{L^2} \\
& \leq C 2^{-k_1/2} \cdot k^3 \cdot (2^{(J_0-2k_1)/2} + 1)^{-1} \cdot 2^{-(l_1-k_1)} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2+Y_{k_2}}} \\
& \leq C 2^{-(l_1-k_1)/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2+Y_{k_2}}}.
\end{aligned} \tag{7.11}$$

Finally, we prove that

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, l_2, \leq J_0}^+) \right\|_{Y_k} \\
& \leq C 2^{-(l_1-k_1)/8} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2+Y_{k_2}}}.
\end{aligned} \tag{7.12}$$

Recall that (see (6.5))

$$\begin{aligned}
& \Omega[(\xi_1, \mu_1), (\xi_2, \mu_2)] = -\omega(\xi_1 + \xi_2, \mu_1 + \mu_2) + \omega(\xi_1, \mu_1) + \omega(\xi_2, \mu_2) = -\frac{\xi_1 \xi_2}{\xi_1 + \xi_2} \\
& \times [(\sqrt{3}\xi_1 - \mu_1/\xi_1) + (\sqrt{3}\xi_2 + \mu_2/\xi_2)] \cdot [(\sqrt{3}\xi_1 + \mu_1/\xi_1) + (\sqrt{3}\xi_2 - \mu_2/\xi_2)].
\end{aligned} \tag{7.13}$$

Thus, for $\xi_2 \in I_{k_2}$, $\mu_2 \in [2^{2k-11}, 2^{2k+11}]$, $\xi_1 \in I_{k_1}$, and $|\mu_1| \leq 2^{k-k_1/2-9}$

$$|\Omega[(\xi_1, \mu_1), (\xi_2, \mu_2)]| \geq 2^{k+k_1-4} |(\sqrt{3}\xi_1 + \mu_1/\xi_1) + (\sqrt{3}\xi_2 - \mu_2/\xi_2)|. \tag{7.14}$$

Let $\varphi : \mathbb{R} \rightarrow [0, 1]$ denote a smooth function supported in $[-1, 1]$ with the property that

$$\sum_{m \in \mathbb{Z}} \varphi(s - m) \equiv 1.$$

Let $\epsilon = 2^{-(l_1+k_1)/2}$. For $m \in \mathbb{Z}$ we define

$$\begin{cases} f_{k_1, l_1, \leq J_0}^{+, m}(\xi_1, \mu_1, \tau_1) = f_{k_1, l_1, \leq J_0}(\xi_1, \mu_1, \tau_1) \cdot \varphi((\sqrt{3}\xi_1 + \mu_1/\xi_1)/\epsilon - m); \\ f_{k_2, l_2, \leq J_0}^{+, m}(\xi_2, \mu_2, \tau_2) = f_{k_2, l_2, \leq J_0}^+(\xi_2, \mu_2, \tau_2) \cdot \varphi((\sqrt{3}\xi_2 - \mu_2/\xi_2)/\epsilon + m). \end{cases} \tag{7.15}$$

The important observation is that, in view of (7.14) and the definition of J_0 ,

$$\eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0}^{+, m} * f_{k_2, l_2, \leq J_0}^{+, m'}) \equiv 0 \text{ unless } |m - m'| \leq 4.$$

Thus, using the definitions and Lemma 2.1 (c),

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, l_2, \leq J_0}^+) \right\|_{Y_k} \\
& \leq \sum_{|m-m'| \leq 4} 2^k \left\| \chi_k(\xi) (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0}^{+,m} * f_{k_2, l_2, \leq J_0}^{+,m'}) \right\|_{Y_k} \\
& \leq C \sum_{|m-m'| \leq 4} 2^{k/2} \|\mathcal{F}^{-1}(f_{k_1, l_1, \leq J_0}^{+,m})\|_{L_y^1 L_{x,t}^\infty} \cdot \|\mathcal{F}^{-1}(f_{k_2, l_2, \leq J_0}^{+,m'})\|_{L_y^\infty L_{x,t}^2}.
\end{aligned} \tag{7.16}$$

We use the elementary bound

$$\|g\|_{L^1(\mathbb{R})}^2 \leq C \|g\|_{L^2(\mathbb{R})} \cdot \|(y+i) \cdot g\|_{L^2(\mathbb{R})} \tag{7.17}$$

for any $g \in L^2(\mathbb{R})$, Lemma 5.4 (b), and the definitions to estimate

$$\begin{aligned}
& \|\mathcal{F}^{-1}(f_{k_1, l_1, \leq J_0}^{+,m})\|_{L_y^1 L_{x,t}^\infty} \leq C 2^{(l_1 - k_1)/4} \left(\sum_{j \leq J_0} 2^{j/2} \|(I - \partial_{\tau_1}^2) f_{k_1, l_1, j}^{+,m}\|_{L^2} \right)^{1/2} \\
& \quad \times \left(\sum_{j \leq J_0} 2^{j/2} \|(I - \partial_{\tau_1}^2)(\partial_{\mu_1} + I) f_{k_1, l_1, j}^{+,m}\|_{L^2} \right)^{1/2} \\
& \leq C 2^{(l_1 - k_1)/4} \|(1 + |\tau_1 - \omega(\xi_1, \mu_1)|)^{1/2+1/40} (I - \partial_{\tau_1}^2) f_{k_1, l_1, \leq J_0}^{+,m}\|_{L^2}^{1/2} \\
& \quad \times \|(1 + |\tau_1 - \omega(\xi_1, \mu_1)|)^{1/2+1/40} (I - \partial_{\tau_1}^2)(\partial_{\mu_1} + I) f_{k_1, l_1, \leq J_0}^{+,m}\|_{L^2}^{1/2},
\end{aligned}$$

where $f_{k_1, l_1, j}^{+,m} = f_{k_1, l_1, j} \cdot \varphi((\sqrt{3}\xi_1 + \mu_1/\xi_1)/\epsilon - m)$. Thus, using Lemma 2.1 (b), with $A = \|(1 + |\tau_1 - \omega(\xi_1, \mu_1)|)^{1/2+1/40} (I - \partial_{\tau_1}^2) f_{k_1, l_1}\|_{L^2}$

$$\begin{aligned}
& \left[\sum_{m \in \mathbb{Z}} \|\mathcal{F}^{-1}(f_{k_1, l_1, \leq J_0}^{+,m})\|_{L_y^1 L_{x,t}^\infty}^2 \right]^{1/2} \\
& \leq C 2^{(l_1 - k_1)/4} \cdot A^{1/2} \\
& \quad \times \left[\|(1 + |\tau_1 - \omega(\xi_1, \mu_1)|)^{1/2+1/40} (I - \partial_{\tau_1}^2)(\partial_{\mu_1} + I) f_{k_1, l_1}\|_{L^2} + 2^{l_1 - k_1} \cdot A \right]^{1/2} \\
& \leq C 2^{(l_1 - k_1)/4} \cdot 2^{-(l_1 - k_1)/2} \cdot (2^{k_1/20} (k_1 + 1)) \|(I - \partial_{\tau_1}^2) f_{k_1, l_1}\|_{V_{k_1}}^{1/2} \\
& \quad \times (2^{k_1/20} (k_1 + 1)) \|(I - \partial_{\tau_1}^2) f_{k_1, l_1}\|_{V_{k_1} \cap W_{k_1}}^{1/2}.
\end{aligned}$$

We substitute this last bound into (7.16) and, using Lemma 5.2 (b), (3.14) and $2k_1 \leq l_1$, we conclude that the right-hand side of (7.16) is dominated by

$$\begin{aligned}
& C 2^{k/2} \left[\sum_{m \in \mathbb{Z}} \|\mathcal{F}^{-1}(f_{k_1, l_1, \leq J_0}^{+,m})\|_{L_y^1 L_{x,t}^\infty}^2 \right]^{1/2} \cdot \left[\sum_{m \in \mathbb{Z}} \|\mathcal{F}^{-1}(f_{k_2, l_2, \leq J_0}^{+,m})\|_{L_y^\infty L_{x,t}^2}^2 \right]^{1/2} \\
& \leq C 2^{-(l_1 - k_1)/8} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned}$$

This gives the bound (7.12). The bound (7.4) follows from the bounds (7.6), (7.9), (7.10), (7.11), and (7.12).

We estimate now the contribution of $f_{k_1, l_1} * f_{k_2, l_2}$, $k + k_1 - 10 \leq l_1 \leq 2k_2 + 12$. In this range we will show that

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k + Y_k} \\ & \leq C 2^{-(l_1 - k_1)/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned} \quad (7.18)$$

Using (2.4), Lemma 5.2 (a), and (7.3) with $J = -1$, we estimate

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\geq k-4}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k} \\ & \leq C 2^k \cdot 2^{-k/2} \|f_{k_1, l_1} * f_{k_2, l_2}\|_{L_{\xi, \mu, \tau}^2} \\ & \leq C 2^k 2^{-k/2} \|\mathcal{F}^{-1}(f_{k_1, l_1})\|_{L_y^2 L_{x, t}^\infty} \cdot \|\mathcal{F}^{-1}(f_{k_2, l_2})\|_{L_y^\infty L_{x, t}^2} \\ & \leq C 2^{-(l_1 - k_1)/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned} \quad (7.19)$$

Using (7.8), Lemma 2.1 (a), (c), and (7.3) we estimate

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq k-5}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2, > k-5}) \right\|_{Y_k} \\ & \leq C 2^{k/2} \cdot \|\mathcal{F}^{-1}(f_{k_1, l_1} * f_{k_2, l_2, > k-5})\|_{L_y^1 L_{x, t}^2} \\ & \leq C 2^{k/2} \|\mathcal{F}^{-1}(f_{k_1, l_1})\|_{L_y^2 L_{x, t}^\infty} \cdot \|\mathcal{F}^{-1}(f_{k_2, l_2, > k-5})\|_{L_y^2 L_{x, t}^2} \\ & \leq C 2^{-(l_1 - k_1)/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned} \quad (7.20)$$

An estimate similar to (7.10) gives

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq k-5}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, > k+2k_1-10} * f_{k_2, l_2, \leq k-5}) \right\|_{Y_k} \\ & \leq C 2^{-(l_1 - k_1)/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned} \quad (7.21)$$

Finally, we use Corollary 6.2 and Lemma 2.1 (b) to estimate

$$\begin{aligned} & 2^k \sum_{j_1=0}^{k+2k_1-10} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq k-5}(\tau - \omega(\xi, \mu)) (f_{k_1, l_1, j_1} * f_{k_2, l_2, \leq k-5}) \right\|_{X_k} \\ & \leq C 2^k \sum_{j_1=0}^{k+2k_1-10} \sum_{j_2=0}^{k-5} 2^{-j/2} \left\| \eta_j(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, j_1} * f_{k_2, l_2, j_2}) \right\|_{L^2} \\ & \leq C 2^k \sum_{j_1=0}^{k+2k_1-10} \sum_{j_2=0}^{k-5} 2^{-(2k+k_1)/2} \cdot 2^{j_1/2} \|f_{k_1, l_1, j_1}\|_{L^2} \cdot 2^{j_2/2} \|f_{k_2, l_2, j_2}\|_{L^2} \\ & \leq C 2^{-k_1/2} \cdot k^3 \cdot 2^{-(l_1 - k_1)} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}} \\ & \leq C 2^{-(l_1 - k_1)/2} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned} \quad (7.22)$$

The bound (7.18) follows from (7.19), (7.20), (7.21), and (7.22).

We estimate now the contribution of $\sum_{l_1 \geq 2k_2+13} f_{k_1, l_1} * f_{k_2, l_2}$: using (2.4) and Lemma 5.5

$$\begin{aligned}
& \left\| \chi_k(\xi) \cdot (2^k + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot \left(\sum_{l_1 \geq 2k_2+13} f_{k_1, l_1} * f_{k_2, l_2} \right) \right\|_{X_k} \\
& \leq C2^{-k} \left\| \chi_k(\xi) \cdot \mu \cdot \left(\sum_{l_1 \geq 2k_2+13} f_{k_1, l_1} * f_{k_2, l_2} \right) \right\|_{L^2} \\
& \leq C2^{-k} \left[\sum_{l_1 \geq 2k_2+13} \|2^{l_1} f_{k_1, l_1} * f_{k_2, l_2}\|_{L^2}^2 \right]^{1/2} \\
& \leq C2^{-k+k_1} \left[\sum_{l_1 \geq 2k_2+13} \|2^{l_1-k_1} f_{k_1, l_1}\|_{X_{k_1+Y_{k_1}}}^2 \cdot \|f_{k_2, l_2}\|_{X_k+Y_k}^2 \right]^{1/2} \\
& \leq C2^{k_1-k} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2}+Y_{k_2}}.
\end{aligned} \tag{7.23}$$

The main bound (7.2) follows from (7.4), (7.18), and (7.23). \square

Lemma 7.3. *With the notation in Proposition 7.1,*

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, 2k_2-10}) \right\|_{V_k \cap W_k} \\
& \leq C(2^{-k_1/8} + 2^{-(k-k_1)/8}) \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, 2k_2-10}\|_{V_{k_2} \cap W_{k_2}}.
\end{aligned}$$

Proof of Lemma 7.3. As in Lemma 7.2, it suffices to prove that

$$\begin{aligned}
& \left\| \chi_k(\xi) \cdot (2^k + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, 2k_2-10}) \right\|_{X_k} \\
& \leq C(2^{-k_1/8} + 2^{-(k-k_1)/8}) \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, 2k_2-10}\|_{X_{k_2}+Y_{k_2}}.
\end{aligned} \tag{7.24}$$

We estimate first the contribution of $f_{k_1, l_1} * f_{k_2, 2k_2-10}$, $l_1 \in [2k_1, 2k+10]$. Let

$$J_0 = 2k + k_1 - 40. \tag{7.25}$$

Using (2.4), (2.12), and Lemma 5.5, we estimate

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\geq 2k-39}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, 2k_2-10}) \right\|_{X_k} \\
& \leq C2^k 2^{-k} \|f_{k_1, l_1} * f_{k_2, 2k_2-10}\|_{L^2} \\
& \leq C \|\mathcal{F}^{-1}(f_{k_1, l_1})\|_{L^4} \cdot \|\mathcal{F}^{-1}(f_{k_2, 2k_2-10})\|_{L^4} \\
& \leq C2^{k_1-l_1} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, 2k_2-10}\|_{X_{k_2}+Y_{k_2}}.
\end{aligned} \tag{7.26}$$

We have the L^∞ bound

$$\begin{aligned}
\|\mathcal{F}^{-1}(f_{k_1, l_1})\|_{L^\infty} & \leq C \sum_{j \geq 0} 2^{j/2} 2^{k_1/2} 2^{l_1/2} \|f_{k_1, l_1, j}\|_{L^2} \\
& \leq C(k_1 + 1) 2^{3k_1/2} 2^{-l_1/2} \|f_{k_1}\|_{V_{k_1}}.
\end{aligned} \tag{7.27}$$

Thus, using (2.4) and (2.12), we estimate

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq 2k-40} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, 2k_2-10, > J_0}) \right\|_{X_k} \\
& \leq C 2^k \left\| f_{k_1, l_1} * f_{k_2, 2k_2-10, > J_0} \right\|_{L^2} \\
& \leq C 2^k \left\| \mathcal{F}^{-1}(f_{k_1, l_1}) \right\|_{L^\infty} \cdot \left\| f_{k_2, 2k_2-10, > J_0} \right\|_{L^2} \\
& \leq C(k_1 + 1) 2^{(k_1 - l_1)/2} \left\| f_{k_1} \right\|_{V_{k_1}} \cdot \left\| f_{k_2, 2k_2-10} \right\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{7.28}$$

As in the proof of Lemma 5.5 (see (5.25)) we have

$$\left\| \mathcal{F}^{-1}(f_{k_1, l_1, > J_0}) \right\|_{L^4} \leq C \sum_{j \geq J_0+1} 2^{j/2} \left\| f_{k_1, l_1, j} \right\|_{L^2} \leq C 2^{-(2k-k_1)/2} 2^{k_1-l_1} \left\| f_{k_1} \right\|_{V_{k_1}}.$$

Thus, using (2.4) and (2.12) and Lemma 5.5,

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq 2k-40} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, > J_0} * f_{k_2, 2k_2-10, \leq J_0}) \right\|_{X_k} \\
& \leq C 2^k \left\| f_{k_1, l_1, > J_0} * f_{k_2, 2k_2-10, \leq J_0} \right\|_{L^2} \\
& \leq C 2^k \left\| \mathcal{F}^{-1}(f_{k_1, l_1, > J_0}) \right\|_{L^4} \cdot \left\| \mathcal{F}^{-1}(f_{k_2, 2k_2-10, \leq J_0}) \right\|_{L^4} \\
& \leq C 2^{3k_1/2-l_1} \left\| f_{k_1} \right\|_{V_{k_1}} \cdot \left\| f_{k_2, 2k_2-10} \right\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{7.29}$$

Finally, we observe that

$$\eta_{\leq 2k-40} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, 2k_2-10, \leq J_0}) \equiv 0,$$

unless $l_1 \in [k + k_1 - 10, k + k_1 + 10]$, which is a consequence of the identity (7.13).

Using Corollary 6.2, Lemma 2.1 (b), and the definitions, we estimate

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq 2k-40} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, 2k_2-10, \leq J_0}) \right\|_{X_k} \\
& \leq C 2^k \sum_{j_1, j_2=0}^{J_0} \sum_{j=0}^{2k-40} 2^{-j/2} \left\| \eta_j (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, j_1} * f_{k_2, 2k_2-10, j_2}) \right\|_{L^2} \\
& \leq C 2^k \sum_{j, j_1, j_2=0}^{J_0} 2^{-(2k+k_1)/2} \cdot 2^{j_1/2} \left\| f_{k_1, l_1, j_1} \right\|_{L^2} \cdot 2^{j_2/2} \left\| f_{k_2, 2k_2-10, j_2} \right\|_{L^2} \\
& \leq C 2^{-k_1/2} \cdot k^3 \cdot 2^{-(l_1-k_1)} \left\| f_{k_1} \right\|_{V_{k_1}} \cdot \left\| f_{k_2, 2k_2-10} \right\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{7.30}$$

Thus, using (7.26), (7.28), (7.29), and (7.30) with $l_1 \in [k + k_1 - 10, k + k_1 + 10]$, we have

$$\begin{aligned}
& \sum_{l_1=2k_1}^{2k+10} 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1, l_1} * f_{k_2, 2k_2-10}) \right\|_{X_k} \\
& \leq C 2^{-k_1/4} \left\| f_{k_1} \right\|_{V_{k_1}} \cdot \left\| f_{k_2, 2k_2-10} \right\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{7.31}$$

We estimate now the contribution of $\sum_{l_1 \geq 2k+11} f_{k_1, l_1} * f_{k_2, 2k_2-10}$: using (2.4) and Lemma 5.5, we estimate as in (7.23)

$$\begin{aligned} & \left\| \chi_k(\xi) \cdot (2^k + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot \left(\sum_{l_1 \geq 2k+11} f_{k_1, l_1} * f_{k_2, 2k_2-10} \right) \right\|_{X_k} \\ & \leq C 2^{k_1-k} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, 2k_2-10}\|_{X_{k_2}+Y_{k_2}}. \end{aligned} \quad (7.32)$$

The main bound (7.24) follows from (7.31) and (7.32). \square

Lemma 7.4. *With the notation in Proposition 7.1, for any $l_2 \geq 2k_2 + 10$*

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, l_2}) \right\|_{V_k \cap W_k} \\ & \leq C 2^{-(l_2-2k_2)/4} (2^{-k_1/8} + 2^{-(k-k_1)/8}) \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{V_{k_2} \cap W_{k_2}}. \end{aligned}$$

Proof of Lemma 7.4. As in Lemma 7.2, it suffices to prove that

$$\begin{aligned} & \left\| \chi_k(\xi) \cdot (2^{l_2-k} + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, l_2}) \right\|_{X_k} \\ & \leq C 2^{(l_2-2k_2) \cdot (3/4)} (2^{-k_1/8} + 2^{-(k-k_1)/8}) \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2}+Y_{k_2}}. \end{aligned} \quad (7.33)$$

We estimate first the contribution of $f_{k_1, l_1} * f_{k_2, l_2}$, for

$$l_1 \in [2k_1, l_2 + 10] \setminus [l_2 - k_2 + k_1 - 10, l_2 - k_2 + k_1 + 10]. \quad (7.34)$$

Let

$$J_0 = l_2 + k_1 - 40. \quad (7.35)$$

Using (2.4), (7.3), and Lemma 5.2, we estimate

$$\begin{aligned} & 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\geq J_0+1}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k} \\ & \leq C 2^{l_2-k} 2^{-k} \|f_{k_1, l_1} * f_{k_2, l_2}\|_{L^2} \\ & \leq C 2^{l_2-2k_2} \|\mathcal{F}^{-1}(f_{k_1, l_1})\|_{L_y^2 L_{x,t}^\infty} \cdot \|\mathcal{F}^{-1}(f_{k_2, l_2})\|_{L_y^\infty L_{x,t}^2} \\ & \leq C 2^{(l_2-2k_2)/2} 2^{-(l_1-k_1)/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2}+Y_{k_2}}. \end{aligned} \quad (7.36)$$

Using the L^∞ bound (7.27), (2.4), and (2.12), we estimate

$$\begin{aligned} & 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2, > J_0}) \right\|_{X_k} \\ & \leq C 2^{l_2-k} \|f_{k_1, l_1} * f_{k_2, l_2, > J_0}\|_{L^2} \\ & \leq C 2^{l_2-k} \|\mathcal{F}^{-1}(f_{k_1, l_1})\|_{L^\infty} \cdot \|f_{k_2, l_2, > J_0}\|_{L^2} \\ & \leq C(k_1 + 1) 2^{(k_1-l_1)/2} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2}+Y_{k_2}}. \end{aligned} \quad (7.37)$$

Using (2.4), (7.3), and Lemma 5.2,

$$\begin{aligned}
& 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, > J_0} * f_{k_2, l_2, \leq J_0}) \right\|_{X_k} \\
& \leq C 2^{l_2-k} \left\| f_{k_1, l_1, > J_0} * f_{k_2, l_2, \leq J_0} \right\|_{L^2} \\
& \leq C 2^{l_2-k} \left\| \mathcal{F}^{-1}(f_{k_1, l_1, > J_0}) \right\|_{L_y^2 L_{x,t}^\infty} \cdot \left\| \mathcal{F}^{-1}(f_{k_2, l_2, \leq J_0}) \right\|_{L_y^\infty L_{x,t}^2} \\
& \leq C 2^{-(l_1-k_1)/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{7.38}$$

Finally, we observe that for l_1 as in (7.34)

$$\eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, l_2, \leq J_0}) \equiv 0,$$

which is a consequence of the identity (7.13). Thus, for l_1 as in (7.34),

$$\begin{aligned}
& 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k} \\
& \leq C 2^{(l_2-2k_2)/2} 2^{-(l_1-k_1)/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{7.39}$$

We estimate now the contribution of $f_{k_1, l_1} * f_{k_2, l_2}$, for

$$l_1 \in [l_2 - k_2 + k_1 - 10, l_2 - k_2 + k_1 + 10]. \tag{7.40}$$

Let

$$J_1 = 2k + k_1 - 40.$$

As in (7.36), (7.37), and (7.38), using also $2^{k_1-l_1} \approx 2^{k_2-l_2}$, we estimate

$$\begin{aligned}
& 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\geq 2k-39}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k} \\
& + 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq 2k-40}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2, > J_1}) \right\|_{X_k} \\
& + 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq 2k-40}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, > J_1} * f_{k_2, l_2, \leq J_1}) \right\|_{X_k} \\
& \leq C 2^{(l_2-2k_2)/2} 2^{-k/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{7.41}$$

In addition, using Corollary 6.2, Lemma 2.1 (b), and the definitions, we estimate

$$\begin{aligned}
& 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq 2k-40}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_1} * f_{k_2, l_2, \leq J_1}) \right\|_{X_k} \\
& \leq C 2^{l_2-k} \sum_{j_1, j_2=0}^{J_1} \sum_{j=0}^{2k-40} 2^{-j/2} \left\| \eta_j(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, j_1} * f_{k_2, l_2, j_2}) \right\|_{L^2} \\
& \leq C 2^{l_2-k} \sum_{j, j_1, j_2=0}^{J_1} 2^{-(2k+k_1)/2} \cdot 2^{j_1/2} \|f_{k_1, l_1, j_1}\|_{L^2} \cdot 2^{j_2/2} \|f_{k_2, l_2, j_2}\|_{L^2} \\
& \leq C 2^{-k_1/2} \cdot k^3 \cdot 2^{-k} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{7.42}$$

We estimate now the contribution of $\sum_{l_1 \geq l_2 + 11} f_{k_1, l_1} * f_{k_2, l_2}$: using (2.4) and Lemma 5.5, we estimate as in (7.23)

$$\begin{aligned}
& \left\| \chi_k(\xi) \cdot (2^{l_2 - k} + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot \left(\sum_{l_1 \geq l_2 + 11} f_{k_1, l_1} * f_{k_2, l_2} \right) \right\|_{X_k} \\
& \leq C 2^{-k} \left\| \mu \cdot \left(\sum_{l_1 \geq l_2 + 11} f_{k_1, l_1} * f_{k_2, l_2} \right) \right\|_{L^2} \\
& \leq C 2^{k_1 - k} \left[\sum_{l_1 \geq l_2 + 11} \|2^{l_1 - k_1} f_{k_1, l_1} * f_{k_2, l_2}\|_{L^2}^2 \right]^{1/2} \\
& \leq C 2^{k_1 - k} \left[\sum_{l_1 \geq l_2 + 11} \|2^{l_1 - k_1} f_{k_1, l_1}\|_{X_{k_1} + Y_{k_1}}^2 \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}^2 \right]^{1/2} \\
& \leq C 2^{k_1 - k} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{7.43}$$

The main bound (7.33) follows from (7.39), (7.41), (7.42), and (7.43). \square

8. DYADIC BILINEAR ESTIMATES II

In this section we prove the bound (4.1) for $k \geq 40$ and $k_1 \leq 0$.

Proposition 8.1. *Assume $k \geq 40$, $k_2 \in [k - 2, k + 2]$, $k_1 \leq 0$, $f_{k_1} \in V_{k_1} \cap W_{k_1}$, $f_{k_2} \in V_{k_2} \cap W_{k_2}$, and $\mathcal{F}^{-1}(f_{k_1})$ is supported in $\mathbb{R}^2 \times [-2, 2]$. Then*

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2}) \right\|_{V_k \cap W_k} \\
& \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \|f_{k_2}\|_{V_{k_2} \cap W_{k_2}}.
\end{aligned} \tag{8.1}$$

Proposition 8.1 follows from Lemma 8.2, Lemma 8.3, and Lemma 8.4 below. We start by decomposing⁴

$$f_{k_2} = f_{k_2, 2k_2 - 10} + \sum_{l_2 \geq 2k_2 - 9} f_{k_2, l_2} = f_{k_2} \cdot \eta_{\leq 2k_2 - 10}(\mu_2) + \sum_{l_2 \geq 2k_2 - 9} f_{k_2} \cdot \eta_{l_2}(\mu_2).$$

and

$$f_{k_1} = f_{k_1, k_1} + \sum_{l_1 \geq k_1 + 1} f_{k_1, l_1} = f_{k_1} \cdot \eta_0(\mu_1/2^{k_1}) + \sum_{l_1 \geq k_1 + 1} f_{k_1} \cdot \eta_{l_1}(\mu_1).$$

⁴In the decomposition below we make an abuse of notation when we write that $f_{k_2, 2k_2} = \sum_{l_i < 2k_2 + 1} f_{k_2, l_2}$ and $f_{k_1, k_1} = \sum_{l_1 < k_1 + 1} f_{k_1, l_1}$. One can see in the rest of the paper that this notation avoids some unnecessary technicalities. One example of its efficiency is in the fact that for any $k_1 \leq 0$, $l_1 < k_1 + 1$

$$(1 + |\xi| + |\mu/\xi|) |f_{k_1, l_1}| \sim |f_{k_2, l_2}|$$

and hence we can simply write

$$(1 + |\xi| + |\mu/\xi|) |f_{k_1, k_1}| \sim |f_{k_1, k_1}|.$$

Our notation also explains why in the proof of the lemmas below we will always assume that $l_1 \geq k_1$.

For any $J \in \mathbb{Z}$ let $f_{k_i, l_i, J} = f_{k_i, l_i} \cdot \eta_J(\tau - \omega(\xi, \mu))$, $f_{k_i, l_i, \leq J} = f_{k_i, l_i} \cdot \eta_{\leq J}(\tau - \omega(\xi, \mu))$, and $f_{k_i, l_i, > J} = f_{k_i, l_i} \cdot \eta_{\geq J+1}(\tau - \omega(\xi, \mu))$, $i = 1, 2$.

Lemma 8.2. *With the notation in Proposition 8.1, for any $l_2 \in [2k_2 - 9, 2k_2 + 9]$*

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, l_2}) \right\|_{V_k \cap W_k} \\ & \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \|f_{k_2, l_2}\|_{V_{k_2} \cap W_{k_2}}. \end{aligned}$$

Proof of Lemma 8.2. In view of the definitions and Lemma 2.1 (b), it suffices to prove that

$$\begin{aligned} & \left\| \chi_k(\xi) \cdot (2^{2k} + i\mu)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, l_2}) \right\|_{X_k + Y_k} \\ & + 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * (\partial_\mu + I)f_{k_2, l_2}) \right\|_{X_k + Y_k} \\ & \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot (2^k \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}} + \|(\partial_\mu + I)f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}). \end{aligned} \quad (8.2)$$

For this, it suffices to prove that

$$\begin{aligned} & \left\| \chi_k(\xi) \cdot (2^k + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, l_2}) \right\|_{X_k + Y_k} \\ & \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned} \quad (8.3)$$

In view of Lemma 2.1 (a), Lemma 5.3 (a), (3.14), and the support assumption on $\mathcal{F}^{-1}(f_{k_1})$,

$$\begin{cases} \|\mathcal{F}^{-1}(\sum_{l_1=k_1}^0 f_{k_1, l_1})\|_{L_y^2 L_{x,t}^\infty} \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1}}; \\ \|\mathcal{F}^{-1}(f_{k_1, l_1})\|_{L_y^2 L_{x,t}^\infty} \leq C 2^{(2l_1+k_1)/4} \cdot 2^{k_1-l_1} \|f_{k_1}\|_{V_{k_1}} \text{ for } l_1 \geq 1. \end{cases} \quad (8.4)$$

Also, using the elementary inequality (7.17),

$$\|\mathcal{F}^{-1}(\sum_{l_1=k_1}^0 f_{k_1, l_1})\|_{L_y^1 L_{x,t}^\infty} \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}}. \quad (8.5)$$

We start by estimating the contribution of $\sum_{l_1=k_1}^0 f_{k_1, l_1} * f_{k_2, l_2}$. Using the definitions, Lemma 2.1 (a), (c), Lemma 5.2 (a), (8.4), and (8.5), we estimate

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq k-10}(\tau - \omega(\xi, \mu)) \cdot \left(\sum_{l_1=k_1}^0 f_{k_1, l_1} * f_{k_2, l_2} \right) \right\|_{Y_k} \\ & \leq C 2^{k/2} \left\| \mathcal{F}^{-1} \left(\sum_{l_1=k_1}^0 f_{k_1, l_1} * f_{k_2, l_2} \right) \right\|_{L_y^1 L_{x,t}^2} \\ & \leq C 2^{k/2} \left\| \mathcal{F}^{-1} \left(\sum_{l_1=k_1}^0 f_{k_1, l_1} \right) \right\|_{L_y^1 L_{x,t}^\infty} \cdot \left\| \mathcal{F}^{-1}(f_{k_2, l_2}) \right\|_{L_y^\infty L_{x,t}^2} \\ & \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned}$$

In addition, using (2.4),

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\geq k-9}(\tau - \omega(\xi, \mu)) \cdot \left(\sum_{l_1=k_1}^0 f_{k_1, l_1} * f_{k_2, l_2} \right) \right\|_{X_k} \\
& \leq C 2^k \cdot 2^{-k/2} \left\| \sum_{l_1=k_1}^0 f_{k_1, l_1} * f_{k_2, l_2} \right\|_{L_{\xi, \mu, \tau}^2} \\
& \leq C 2^{k/2} \left\| \mathcal{F}^{-1} \left(\sum_{l_1=k_1}^0 f_{k_1, l_1} \right) \right\|_{L_y^2 L_{x, t}^\infty} \cdot \left\| \mathcal{F}^{-1}(f_{k_2, l_2}) \right\|_{L_y^\infty L_{x, t}^2} \\
& \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned}$$

Thus

$$\begin{aligned}
& \left\| \chi_k(\xi) \cdot (2^k + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot \left(\sum_{l_1=k_1}^0 f_{k_1, l_1} * f_{k_2, l_2} \right) \right\|_{X_k + Y_k} \\
& \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{8.6}$$

We estimate now the contribution of $f_{k_1, l_1} * f_{k_2, l_2}$, $1 \leq l_1 \leq k + 2k_1 - 10$. In this range we will show that

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k + Y_k} \\
& \leq C 2^{3k_1/4} 2^{-l_1/4} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{8.7}$$

Let

$$J_0 \text{ denote the smallest integer } \geq k + k_1 - l_1/2 - 10.$$

Using (2.4), Lemma 2.1 (a), Lemma 5.2 (a), and (8.4) we estimate

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\geq J_0+1}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k} \\
& \leq C 2^k \cdot 2^{-J_0/2} \|f_{k_1, l_1} * f_{k_2, l_2}\|_{L_{\xi, \mu, \tau}^2} \\
& \leq C 2^k 2^{-J_0/2} \left\| \mathcal{F}^{-1}(f_{k_1, l_1}) \right\|_{L_y^2 L_{x, t}^\infty} \cdot \left\| \mathcal{F}^{-1}(f_{k_2, l_2}) \right\|_{L_y^\infty L_{x, t}^2} \\
& \leq C 2^{3k_1/4} 2^{-l_1/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{8.8}$$

As in (7.7), we decompose

$$\begin{aligned}
f_{k_2, l_2} &= f_{k_2, l_2, \leq J_0}^+ + f_{k_2, l_2, \leq J_0}^- + f_{k_2, l_2, > J_0} = f_{k_2, l_2} \cdot \eta_{\leq J_0}(\tau_2 - \omega(\xi_2, \mu_2)) \mathbf{1}_+(\mu_2) \\
&\quad + f_{k_2, l_2} \cdot \eta_{\leq J_0}(\tau_2 - \omega(\xi_2, \mu_2)) \mathbf{1}_-(\mu_2) \\
&\quad + f_{k_2, l_2} \cdot \eta_{\geq J_0+1}(\tau_2 - \omega(\xi_2, \mu_2)).
\end{aligned}$$

Using (2.12),

$$\|f_{k_2, l_2, > J_0}\|_{L^2} \leq C 2^{-J_0/2} \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.$$

Thus, using the definitions, Lemma 2.1 (a), (c), and (8.4) we estimate

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2, > J_0}) \right\|_{Y_k} \\
& \leq C 2^{k/2} \cdot \left\| \mathcal{F}^{-1}(f_{k_1, l_1} * f_{k_2, l_2, > J_0}) \right\|_{L_y^1 L_{x, t}^2} \\
& \leq C 2^{k/2} \left\| \mathcal{F}^{-1}(f_{k_1, l_1}) \right\|_{L_y^2 L_{x, t}^\infty} \cdot \left\| \mathcal{F}^{-1}(f_{k_2, l_2, > J_0}) \right\|_{L_y^2 L_{x, t}^2} \\
& \leq C 2^{3k_1/4} 2^{-l_1/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{8.9}$$

Also, using Lemma 2.1(a), Lemma 5.3 (a), and the definitions

$$\left\| \mathcal{F}^{-1}(f_{k_1, l_1, > J_0}) \right\|_{L_y^2 L_{x, t}^\infty} \leq C 2^{-J_0/2} 2^{(2l_1 + k_1)/4} \cdot 2^{k_1 - l_1} \|f_{k_1}\|_{V_{k_1}}.$$

An estimate similar to (8.9) then gives

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, > J_0} * f_{k_2, l_2, \leq J_0}^\pm) \right\|_{Y_k} \\
& \leq C 2^{3k_1/4} 2^{-l_1/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{8.10}$$

It remains to estimate $2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, l_2, \leq J_0}^\pm) \right\|_{Y_k}$. We will use Lemma 5.2 (b) and Lemma 5.3 (b) to exploit some additional orthogonality. By symmetry, it suffices to prove that

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, l_2, \leq J_0}^+) \right\|_{Y_k} \\
& \leq C 2^{k_1} 2^{-l_1/4} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{8.11}$$

For $(\xi_1, \mu_1), (\xi_2, \mu_2) \in \mathbb{R}^2$ recall that (see (7.13))

$$\begin{aligned}
& \Omega[(\xi_1, \mu_1), (\xi_2, \mu_2)] = -\omega(\xi_1 + \xi_2, \mu_1 + \mu_2) + \omega(\xi_1, \mu_1) + \omega(\xi_2, \mu_2) = -\frac{\xi_1 \xi_2}{\xi_1 + \xi_2} \\
& \times [(\sqrt{3}\xi_1 - \mu_1/\xi_1) + (\sqrt{3}\xi_2 + \mu_2/\xi_2)] \cdot [(\sqrt{3}\xi_1 + \mu_1/\xi_1) + (\sqrt{3}\xi_2 - \mu_2/\xi_2)].
\end{aligned}$$

Thus, for $\xi_2 \in I_{k_2}$, $\mu_2 \in [2^{2k-11}, 2^{2k+11}]$, $\xi_1 \in I_{k_1}$, and $|\mu_1| \leq 2^{k+2k_1-9}$

$$|\Omega[(\xi_1, \mu_1), (\xi_2, \mu_2)]| \geq 2^{k+k_1-4} |(\sqrt{3}\xi_1 + \mu_1/\xi_1) + (\sqrt{3}\xi_2 - \mu_2/\xi_2)|. \tag{8.12}$$

Let $\varphi : \mathbb{R} \rightarrow [0, 1]$ denote a smooth function supported in $[-1, 1]$ with the property that

$$\sum_{m \in \mathbb{Z}} \varphi(s - m) \equiv 1.$$

Let $\epsilon = 2^{-l_1/2}$. For $m \in \mathbb{Z}$ we define

$$\begin{cases} f_{k_1, l_1, \leq J_0}^{+, m}(\xi_1, \mu_1, \tau_1) = f_{k_1, l_1, \leq J_0}(\xi_1, \mu_1, \tau_1) \cdot \varphi((\sqrt{3}\xi_1 + \mu_1/\xi_1)/\epsilon - m); \\ f_{k_2, l_2, \leq J_0}^{+, m}(\xi_2, \mu_2, \tau_2) = f_{k_2, l_2, \leq J_0}^+(\xi_2, \mu_2, \tau_2) \cdot \varphi((\sqrt{3}\xi_2 - \mu_2/\xi_2)/\epsilon + m). \end{cases} \tag{8.13}$$

The important observation is that, in view of (8.12) and the definition of J_0 ,

$$\eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0}^{+, m} * f_{k_2, l_2, \leq J_0}^{+, m'}) \equiv 0 \text{ unless } |m - m'| \leq 4.$$

Thus, using the definitions and Lemma 2.1 (c),

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, l_2, \leq J_0}^+) \right\|_{Y_k} \\
& \leq \sum_{|m-m'| \leq 4} 2^k \left\| \chi_k(\xi) (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0}^{+,m} * f_{k_2, l_2, \leq J_0}^{+,m'}) \right\|_{Y_k} \\
& \leq C \sum_{|m-m'| \leq 4} 2^{k/2} \|\mathcal{F}^{-1}(f_{k_1, l_1, \leq J_0}^{+,m})\|_{L_y^1 L_{x,t}^\infty} \cdot \|\mathcal{F}^{-1}(f_{k_2, l_2, \leq J_0}^{+,m'})\|_{L_y^\infty L_{x,t}^2}.
\end{aligned} \tag{8.14}$$

Using the bound (7.17), Lemma 5.3 (b), and the definitions

$$\begin{aligned}
\|\mathcal{F}^{-1}(f_{k_1, l_1, \leq J_0}^{+,m})\|_{L_y^1 L_{x,t}^\infty} & \leq C 2^{l_1/4} 2^{k_1/2} \|(\tau_1 - \omega(\xi_1, \mu_1) + i) \cdot (I - \partial_{\tau_1}^2) f_{k_1, l_1, \leq J_0}^{+,m}\|_{L^2}^{1/2} \\
& \quad \times \|(\tau_1 - \omega(\xi_1, \mu_1) + i) \cdot (I - \partial_{\tau_1}^2)(\partial_{\mu_1} + I) f_{k_1, l_1, \leq J_0}^{+,m}\|_{L^2}^{1/2}.
\end{aligned}$$

Thus

$$\begin{aligned}
& \left[\sum_{m \in \mathbb{Z}} \|\mathcal{F}^{-1}(f_{k_1, l_1, \leq J_0}^{+,m})\|_{L_y^1 L_{x,t}^\infty}^2 \right]^{1/2} \\
& \leq C 2^{l_1/4} 2^{k_1/2} 2^{-(l_1 - k_1)/2} \|(I - \partial_\tau^2) f_{k_1, l_1}\|_{V_{k_1}}^{1/2} \cdot \|(I - \partial_\tau^2) f_{k_1, l_1}\|_{V_{k_1} \cap W_{k_1}}^{1/2}.
\end{aligned}$$

We substitute this last bound into (8.14) and, using Lemma 5.2 (b) and (3.14), we conclude that the right-hand side of (8.14) is dominated by

$$\begin{aligned}
& C 2^{k/2} \left[\sum_{m \in \mathbb{Z}} \|\mathcal{F}^{-1}(f_{k_1, l_1, \leq J_0}^{+,m})\|_{L_y^1 L_{x,t}^\infty}^2 \right]^{1/2} \cdot \left[\sum_{m \in \mathbb{Z}} \|\mathcal{F}^{-1}(f_{k_2, l_2, \leq J_0}^{+,m})\|_{L_y^\infty L_{x,t}^2}^2 \right]^{1/2} \\
& \leq C 2^{-l_1/4} 2^{k_1} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned}$$

This gives the bound (8.11). The bound (8.7) follows from the bounds (8.8), (8.9), (8.10), and (8.11).

We estimate now the contribution of $f_{k_1, l_1} * f_{k_2, l_2}$, $l_1 \in [k + 2k_1 - 10, 3k] \cap [1, \infty)$: using (2.4), Lemma 2.1 (a), Lemma 5.2 (a), and (8.4),

$$\begin{aligned}
& \left\| \chi_k(\xi) \cdot (2^k + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k} \\
& \leq C(2^k + 2^{l_1 - k}) \cdot \|f_{k_1, l_1} * f_{k_2, l_2}\|_{L_{\xi, \mu, \tau}^2} \\
& \leq C(2^k + 2^{l_1 - k}) \cdot \|\mathcal{F}^{-1}(f_{k_1, l_1})\|_{L_y^2 L_{x,t}^\infty} \cdot \|\mathcal{F}^{-1}(f_{k_2, l_2})\|_{L_y^\infty L_{x,t}^2} \\
& \leq C 2^{k_1/4} (2^{-(l_1 - k - 2k_1)/2} + 2^{-(3k - l_1)/2}) \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{8.15}$$

Finally, we estimate the contribution of $\sum_{l_1 \geq 3k} f_{k_1, l_1} * f_{k_2, l_2}$: using (2.4) and Lemma 5.5

$$\begin{aligned}
& \left\| \chi_k(\xi) \cdot (2^k + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot \left(\sum_{l_1 \geq 3k} f_{k_1, l_1} * f_{k_2, l_2} \right) \right\|_{X_k} \\
& \leq C 2^{-k} \left\| \chi_k(\xi) \cdot \mu \cdot \left(\sum_{l_1 \geq 3k} f_{k_1, l_1} * f_{k_2, l_2} \right) \right\|_{L^2} \\
& \leq C 2^{-k} \left[\sum_{l_1 \geq 3k} \|2^{l_1} f_{k_1, l_1} * f_{k_2, l_2}\|_{L^2}^2 \right]^{1/2} \\
& \leq C \left[\sum_{l_1 \geq 3k} \|2^{l_1-k} f_{k_1, l_1}\|_{X_{k_1}}^2 \cdot \|f_{k_2, l_2}\|_{X_k + Y_k}^2 \right]^{1/2} \\
& \leq C 2^{k_1-k} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{8.16}$$

The main bound (8.3) follows from (8.6), (8.7), (8.15), and (8.16). \square

Lemma 8.3. *With the notation in Proposition 8.1,*

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, 2k_2-10}) \right\|_{V_k \cap W_k} \\
& \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, 2k_2-10}\|_{V_{k_2} \cap W_{k_2}}.
\end{aligned}$$

Proof of Lemma 8.3. As in Lemma 8.2, it suffices to prove that

$$\begin{aligned}
& \left\| \chi_k(\xi) \cdot (2^k + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, 2k_2-10}) \right\|_{X_k} \\
& \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, 2k_2-10}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{8.17}$$

We estimate first the contribution of $f_{k_1, l_1} * f_{k_2, 2k_2-10}$, $l_1 \in [k_1, 2k+10] \cap \mathbb{Z}$. Using (2.12), for any $J \in \mathbb{Z} \cap [-1, \infty)$,

$$\begin{aligned}
\|\mathcal{F}^{-1}(f_{k_1, l_1, > J})\|_{L^\infty} & \leq C \sum_{j > J} 2^{j/2} 2^{k_1/2} 2^{l_1/2} \|f_{k_1, l_1, j}\|_{L^2} \\
& \leq C 2^{-J/2} 2^{k_1/2} 2^{l_1/2} \|f_{k_1, l_1}\|_{X_{k_1}} \\
& \leq C 2^{-J/2} 2^{3k_1/2} 2^{-l_1/2} \|f_{k_1}\|_{V_{k_1}}.
\end{aligned} \tag{8.18}$$

Let

$$J_0 = 2k + k_1 - 40. \tag{8.19}$$

Using (2.4), (2.12), and (8.18) (with $J = -1$), we estimate

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\geq J_0+1}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, 2k_2-10}) \right\|_{X_k} \\
& \leq C 2^k 2^{-J_0/2} \|f_{k_1, l_1} * f_{k_2, 2k_2-10}\|_{L^2} \\
& \leq C 2^k 2^{-J_0/2} \cdot \|\mathcal{F}^{-1}(f_{k_1, l_1})\|_{L^\infty} \cdot \|f_{k_2, 2k_2-10}\|_{L^2} \\
& \leq C 2^{k_1-l_1/2} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, 2k_2-10}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{8.20}$$

Similarly, using (2.4), (2.12), and (8.18), we estimate

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, 2k_2-10, > J_0}) \right\|_{X_k} \\
& + 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, > J_0} * f_{k_2, 2k_2-10}) \right\|_{X_k} \\
& \leq C 2^{k_1-l_1/2} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, 2k_2-10}\|_{X_{k_2+Y_{k_2}}}.
\end{aligned} \tag{8.21}$$

We observe now that

$$\eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, 2k_2-10, \leq J_0}) \equiv 0,$$

unless $l_1 \in [k + k_1 - 10, k + k_1 + 10] \cap \mathbb{Z}$, which is a consequence of the identity (7.13). As in (8.20) and (8.21), we estimate

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, 2k_2-10, \leq J_0}) \right\|_{X_k} \\
& \leq C 2^k \|f_{k_1, l_1, \leq J_0} * f_{k_2, \leq 2k_2-10, \leq J_0}\|_{L^2} \\
& \leq C 2^k 2^{3k_1/2} 2^{-l_1/2} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, 2k_2-10}\|_{X_{k_2+Y_{k_2}}}.
\end{aligned} \tag{8.22}$$

Using Corollary 6.2, Lemma 2.1 (b), and the definitions, we estimate

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, 2k_2-10, \leq J_0}) \right\|_{X_k} \\
& \leq C 2^k \sum_{j, j_1, j_2=0}^{J_0} 2^{-j/2} \left\| \eta_j(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, j_1} * f_{k_2, 2k_2-10, j_2}) \right\|_{L^2} \\
& \leq C 2^k \sum_{j, j_1, j_2=0}^{J_0} 2^{-(2k+k_1)/2} \cdot 2^{j_1/2} \|f_{k_1, l_1, j_1}\|_{L^2} \cdot 2^{j_2/2} \|f_{k_2, 2k_2-10, j_2}\|_{L^2} \\
& \leq C 2^{-k_1/2} \cdot k^3 \cdot 2^{-(l_1-k_1)} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, 2k_2-10}\|_{X_{k_2+Y_{k_2}}}.
\end{aligned} \tag{8.23}$$

It follows from (8.22) and (8.23) that

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, 2k_2-10, \leq J_0}) \right\|_{X_k} \\
& \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, 2k_2-10}\|_{X_{k_2+Y_{k_2}}},
\end{aligned}$$

for $l_1 \in [k + k_1 - 10, k + k_1 + 10] \cap \mathbb{Z}$. Thus, using also (8.20) and (8.21),

$$\begin{aligned}
& \sum_{l_1=k_1}^{2k+10} 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1, l_1} * f_{k_2, 2k_2-10}) \right\|_{X_k} \\
& \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, 2k_2-10}\|_{X_{k_2+Y_{k_2}}}.
\end{aligned} \tag{8.24}$$

We estimate now the contribution of $\sum_{l_1 \geq 2k+11} f_{k_1, l_1} * f_{k_2, 2k_2-10}$: using (2.4) and Lemma 5.5, we estimate as in (8.16)

$$\begin{aligned} & \left\| \chi_k(\xi) \cdot (2^k + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot \left(\sum_{l_1 \geq 2k+11} f_{k_1, l_1} * f_{k_2, 2k_2-10} \right) \right\|_{X_k} \\ & \leq C 2^{k_1-k} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, \leq 2k_2-10}\|_{X_{k_2}+Y_{k_2}}. \end{aligned} \quad (8.25)$$

The main bound (8.17) follows from (8.24) and (8.25). \square

Lemma 8.4. *With the notation in Proposition 8.1, for any $l_2 \geq 2k_2 + 10$*

$$\begin{aligned} & 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, l_2}) \right\|_{V_k \cap W_k} \\ & \leq C 2^{-(l_2-2k_2)/4} 2^{k_1/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{V_{k_2} \cap W_{k_2}}. \end{aligned}$$

Proof of Lemma 8.4. As in Lemma 7.4, it suffices to prove that

$$\begin{aligned} & \left\| \chi_k(\xi) \cdot (2^{l_2-k} + i\mu/2^k)(\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2, l_2}) \right\|_{X_k} \\ & \leq C 2^{(l_2-2k_2) \cdot (3/4)} 2^{k_1/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2}+Y_{k_2}}. \end{aligned} \quad (8.26)$$

Using Lemma 5.3 and the definitions, for any $J \in [-1, \infty) \cap \mathbb{Z}$, $k_1 \leq 0$, and $l_1 \geq k_1$,

$$\begin{aligned} \|\mathcal{F}^{-1}(f_{k_1, l_1, > J})\|_{L_y^2 L_{x,t}^\infty} & \leq C 2^{-J/2} 2^{k_1/4} (2^{l_1/2} + 1) \cdot 2^{k_1-l_1} \|f_{k_1}\|_{V_{k_1}} \\ & \leq C 2^{-J/2} 2^{k_1/4} 2^{-(l_1-k_1)/2} \|f_{k_1}\|_{V_{k_1}}. \end{aligned} \quad (8.27)$$

Recall also the L^∞ estimate (8.18)

$$\|\mathcal{F}^{-1}(f_{k_1, l_1, > J})\|_{L^\infty} \leq C 2^{-J/2} 2^{k_1} 2^{-(l_1-k_1)/2} \|f_{k_1}\|_{V_{k_1}}. \quad (8.28)$$

We estimate first the contribution of $f_{k_1, l_1} * f_{k_2, l_2}$ for

$$l_1 \in [k_1, l_2 + 10] \setminus [l_2 - k_2 + k_1 - 10, l_2 - k_2 + k_1 + 10]. \quad (8.29)$$

Let

$$J_0 = l_2 + k_1 - 40. \quad (8.30)$$

If $l_2 - 2k_2 + k_1 \geq 0$ then we use (2.4), (8.27), and Lemma 5.2 to estimate

$$\begin{aligned} & 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\geq J_0+1}(\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k} \\ & \leq C 2^{l_2-k} 2^{-k} \|f_{k_1, l_1} * f_{k_2, l_2}\|_{L^2} \\ & \leq C 2^{l_2-2k_2} \|\mathcal{F}^{-1}(f_{k_1, l_1})\|_{L_y^2 L_{x,t}^\infty} \cdot \|\mathcal{F}^{-1}(f_{k_2, l_2})\|_{L_y^\infty L_{x,t}^2} \\ & \leq C 2^{(l_2-2k_2)/2} \cdot 2^{k_1/4} 2^{-(l_1-k_1)/2} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2}+Y_{k_2}}. \end{aligned}$$

If $l_2 - 2k_2 + k_1 \leq 0$ then we use (2.4), (8.28), and (2.12) to estimate

$$\begin{aligned}
& 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\geq J_0+1} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k} \\
& \leq C 2^{l_2-k} 2^{-(l_2+k_1)/2} \left\| f_{k_1, l_1} * f_{k_2, l_2} \right\|_{L^2} \\
& \leq C 2^{l_2-k_2} 2^{-(l_2+k_1)/2} \left\| \mathcal{F}^{-1}(f_{k_1, l_1}) \right\|_{L^\infty} \cdot \left\| f_{k_2, l_2} \right\|_{L^2} \\
& \leq C 2^{(l_2-2k_2)/2} \cdot 2^{k_1/2} 2^{-(l_1-k_1)/2} \left\| f_{k_1} \right\|_{V_{k_1}} \cdot \left\| f_{k_2, l_2} \right\|_{X_{k_2+Y_{k_2}}}.
\end{aligned} \tag{8.31}$$

Thus

$$\begin{aligned}
& 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\geq J_0+1} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k} \\
& \leq C 2^{(l_2-2k_2)/2} \cdot 2^{k_1/4} 2^{-(l_1-k_1)/2} \left\| f_{k_1} \right\|_{V_{k_1}} \cdot \left\| f_{k_2, l_2} \right\|_{X_{k_2+Y_{k_2}}}.
\end{aligned} \tag{8.32}$$

Using the L^∞ bound (8.28), (2.4), and (2.12), we estimate

$$\begin{aligned}
& 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2, > J_0}) \right\|_{X_k} \\
& \leq C 2^{l_2-k} \left\| f_{k_1, l_1} * f_{k_2, l_2, > J_0} \right\|_{L^2} \\
& \leq C 2^{l_2-k} \left\| \mathcal{F}^{-1}(f_{k_1, l_1}) \right\|_{L^\infty} \cdot \left\| f_{k_2, l_2, > J_0} \right\|_{L^2} \\
& \leq C 2^{(l_2-2k_2)/2} \cdot 2^{k_1/2} 2^{-(l_1-k_1)/2} \left\| f_{k_1} \right\|_{V_{k_1}} \cdot \left\| f_{k_2, l_2} \right\|_{X_{k_2+Y_{k_2}}}.
\end{aligned} \tag{8.33}$$

Using (8.28), (2.4), and (2.12),

$$\begin{aligned}
& 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_0} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, > J_0} * f_{k_2, l_2, \leq J_0}) \right\|_{X_k} \\
& \leq C 2^{l_2-k} \left\| f_{k_1, l_1, > J_0} * f_{k_2, l_2, \leq J_0} \right\|_{L^2} \\
& \leq C 2^{l_2-k} \left\| \mathcal{F}^{-1}(f_{k_1, l_1, > J_0}) \right\|_{L^\infty} \cdot \left\| f_{k_2, l_2, \leq J_0} \right\|_{L^2} \\
& \leq C 2^{(l_2-2k_2)/2} \cdot 2^{k_1/2} 2^{-(l_1-k_1)/2} \left\| f_{k_1} \right\|_{V_{k_1}} \cdot \left\| f_{k_2, l_2} \right\|_{X_{k_2+Y_{k_2}}}.
\end{aligned} \tag{8.34}$$

Finally, we observe that for l_1 as in (8.29)

$$\eta_{\leq J_0} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_0} * f_{k_2, l_2, \leq J_0}) \equiv 0,$$

which is a consequence of the identity (7.13). Thus, for l_1 as in (8.29),

$$\begin{aligned}
& 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k} \\
& \leq C 2^{(l_2-2k_2)/2} 2^{k_1/4} 2^{-(l_1-k_1)/2} \left\| f_{k_1} \right\|_{V_{k_1}} \cdot \left\| f_{k_2, l_2} \right\|_{X_{k_2+Y_{k_2}}}.
\end{aligned} \tag{8.35}$$

We estimate now the contribution of $f_{k_1, l_1} * f_{k_2, l_2}$, for

$$l_1 \in [l_2 - k_2 + k_1 - 10, l_2 - k_2 + k_1 + 10].$$

Let

$$J_1 = 2k + 2k_1 - 40.$$

As in (8.31), (8.33), and (8.34), using also $2^{k_1-l_1} \approx 2^{k_2-l_2}$, we estimate

$$\begin{aligned}
& 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\geq J_1+1} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2}) \right\|_{X_k} \\
& + 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_1} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1} * f_{k_2, l_2, > J_1}) \right\|_{X_k} \\
& + 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_1} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, > J_1} * f_{k_2, l_2, \leq J_1}) \right\|_{X_k} \\
& \leq C 2^{l_2-k} (2^{J_1} + 1)^{-1/2} 2^{k_1} 2^{-(l_1-k_1)/2} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}} \\
& \leq C 2^{(l_2-2k_2)/2} 2^{k_1/2} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{8.36}$$

In addition, using Corollary 6.2, Lemma 2.1 (b), and the definitions, we estimate

$$\begin{aligned}
& 2^{l_2-k} \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \eta_{\leq J_1} (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, \leq J_1} * f_{k_2, l_2, \leq J_1}) \right\|_{X_k} \\
& \leq C 2^{l_2-k} \sum_{j, j_1, j_2=0}^{J_1} 2^{-j/2} \left\| \eta_j (\tau - \omega(\xi, \mu)) \cdot (f_{k_1, l_1, j_1} * f_{k_2, l_2, j_2}) \right\|_{L^2} \\
& \leq C 2^{l_2-k} \sum_{j, j_1, j_2=0}^{J_1} 2^{-(2k+k_1)/2} \cdot 2^{j_1/2} \|f_{k_1, l_1, j_1}\|_{L^2} \cdot 2^{j_2/2} \|f_{k_2, l_2, j_2}\|_{L^2} \\
& \leq C 2^{k_1/4} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}},
\end{aligned} \tag{8.37}$$

since we may assume that $k + k_1 \geq 0$ (compare with the definition of J_1).

We estimate now the contribution of $\sum_{l_1 \geq l_2+11} f_{k_1, l_1} * f_{k_2, l_2}$: using (2.4) and Lemma 5.5, we estimate as in (7.43)

$$\begin{aligned}
& \left\| \chi_k(\xi) \cdot (2^{l_2-k} + i\mu/2^k) (\tau - \omega(\xi, \mu) + i)^{-1} \cdot \left(\sum_{l_1 \geq l_2+11} f_{k_1, l_1} * f_{k_2, l_2} \right) \right\|_{X_k} \\
& \leq C 2^{k_1-k} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2, l_2}\|_{X_{k_2} + Y_{k_2}}.
\end{aligned} \tag{8.38}$$

The main bound (8.26) follows from (8.35), (8.36), (8.37), and (8.38). \square

9. DYADIC BILINEAR ESTIMATES III

In this section we prove the bound (4.1) for $k \leq 40$.

Proposition 9.1. *Assume $k \leq 40$, $k_2 \in [k-2, k+2]$, $k_1 \leq k-20$, $f_{k_1} \in V_{k_1} \cap W_{k_1}$, and $f_{k_2} \in V_{k_2} \cap W_{k_2}$. Then*

$$\begin{aligned}
& 2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2}) \right\|_{V_k \cap W_k} \\
& \leq C 2^{k_1/2} \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2}\|_{V_{k_2} \cap W_{k_2}}.
\end{aligned} \tag{9.1}$$

Proof of Proposition 9.1. We show first that

$$2^k \left\| \chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2}) \right\|_{V_k} \leq C 2^{k_1/2} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2}\|_{V_{k_2}}. \tag{9.2}$$

Using (2.4) and the definition (2.6), the left-hand side of (9.2) is dominated by

$$\begin{aligned} & C \|(1 + |\mu|) \cdot \chi_k(\xi) \cdot (f_{k_1} * f_{k_2})\|_{L^2} \\ & \leq C \|(|\mu_1 f_{k_1}|) * |f_{k_2}|\|_{L^2} + C \| |f_{k_1}| * [(1 + |\mu_2|)|f_{k_2}|]\|_{L^2}. \end{aligned} \quad (9.3)$$

We observe now that, for $i = 1, 2$

$$\begin{aligned} \|\mathcal{F}^{-1}(|f_{k_i}|)\|_{L^\infty} & \leq C \sum_{l,j \geq 0} \|f_{k_i} \cdot \eta_l(\mu) \cdot \eta_j(\tau - \omega(\xi, \mu))\|_{L^1} \\ & \leq C \sum_{l,j \geq 0} 2^{(k_i + l + j)/2} \|f_{k_i} \cdot \eta_l(\mu) \cdot \eta_j(\tau - \omega(\xi, \mu))\|_{L^2} \\ & \leq C 2^{k_i/2} \sum_{j \geq 0} 2^{j/2} \|f_{k_i} \cdot (1 + |\mu|) \cdot \eta_j(\tau - \omega(\xi, \mu))\|_{L^2} \\ & \leq C 2^{k_i/2} \|f_{k_i}\|_{V_{k_i}}. \end{aligned} \quad (9.4)$$

Thus, using also (2.12), the right hand side of (9.3) is bounded by

$$\begin{aligned} & C \|\mu_1 \cdot f_{k_1}\|_{L^2} \cdot \|\mathcal{F}^{-1}(|f_{k_2}|)\|_{L^\infty} + C \|\mathcal{F}^{-1}(|f_{k_1}|)\|_{L^\infty} \|(1 + |\mu_2|)f_{k_2}\|_{L^2} \\ & \leq C 2^{k_1} \|(\mu_1/2^{k_1}) \cdot f_{k_1}\|_{X_{k_1}} \|f_{k_2}\|_{V_{k_2}} + C 2^{k_1/2} \|f_{k_1}\|_{V_{k_1}} \|(1 + |\mu_2|)f_{k_2}\|_{X_{k_2}} \\ & \leq C 2^{k_1/2} \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2}\|_{V_{k_2}}. \end{aligned}$$

This completes the proof of (9.2).

We show now that

$$2^k \|\chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2})\|_{W_k} \leq C 2^{k_1/2} \|f_{k_1}\|_{V_{k_1}} \|f_{k_2}\|_{V_{k_2} \cap W_{k_2}}. \quad (9.5)$$

In view of the definition (2.7), the left-hand side of (9.5) is bounded by

$$\begin{aligned} & C 2^k \|\chi_k(\xi) \cdot (\mu/\xi)(\tau - \omega(\xi, \mu) + i)^{-2} \cdot (f_{k_1} * f_{k_2})\|_{X_k} \\ & + C 2^k \|\chi_k(\xi) \cdot (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * (\partial_\mu + I)f_{k_2})\|_{X_k}. \end{aligned} \quad (9.6)$$

The first term in (9.6) is dominated by the left-hand side of (9.3). We use (2.4), (9.4), and (2.12) to estimate the second term in (9.6) by

$$\begin{aligned} C 2^k \|f_{k_1} * (\partial_\mu + I)f_{k_2}\|_{L^2} & \leq C \|\mathcal{F}^{-1}(f_{k_1})\|_{L^\infty} \cdot \|(\partial_\mu + I)f_{k_2}\|_{L^2} \\ & \leq C 2^{k_1/2} \|f_{k_1}\|_{X_{k_1}} \cdot \|(\partial_\mu + I)f_{k_2}\|_{X_{k_2}}, \end{aligned}$$

which suffices for (9.5). \square

10. DYADIC BILINEAR ESTIMATES IV

In this section we prove the bound (4.2).

Proposition 10.1. *Assume $k_1, k_2 \in \mathbb{Z}$, $|k_1 - k_2| \leq 100$, $f_{k_1} \in V_{k_1} \cap W_{k_1}$, and $f_{k_2} \in V_{k_2} \cap W_{k_2}$. Then*

$$\left[\sum_{k \in \mathbb{Z}} \left\| 2^k \chi_k(\xi) (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2}) \right\|_{V_k \cap W_k}^2 \right]^{1/2} \leq C \|f_{k_1}\|_{V_{k_1} \cap W_{k_1}} \cdot \|f_{k_2}\|_{V_{k_2} \cap W_{k_2}}. \quad (10.1)$$

Proof of Proposition 10.1. We show first that

$$\left[\sum_{k \in \mathbb{Z}} \left\| 2^k \chi_k(\xi) (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2}) \right\|_{V_k}^2 \right]^{1/2} \leq C \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2}\|_{V_{k_2}}. \quad (10.2)$$

Using (2.4) and the definition (2.6), the left-hand side of (10.2) is dominated by

$$C \left[\sum_{k \in \mathbb{Z}} \left\| (1 + 2^{2k} + |\mu|) \chi_k(\xi) \cdot (f_{k_1} * f_{k_2}) \right\|_{L^2}^2 \right]^{1/2} \leq C(2^{k_1+k_2} + 1) \|f_{k_1} * f_{k_2}\|_{L^2} + C \|\mu \cdot (f_{k_1} * f_{k_2})\|_{L^2}. \quad (10.3)$$

Using Lemma 5.5, the first term in the right-hand side of (10.3) is dominated by

$$C(2^{k_1} + 1) \|\mathcal{F}^{-1}(f_{k_1})\|_{L^4} \cdot (2^{k_2} + 1) \|\mathcal{F}^{-1}(f_{k_2})\|_{L^4} \leq C \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2}\|_{V_{k_2}}.$$

The second term is bounded by

$$\begin{aligned} & C \|\mathcal{F}^{-1}(\mu_1 \cdot f_{k_1})\|_{L^4} \cdot \|\mathcal{F}^{-1}(f_{k_2})\|_{L^4} + C \|\mathcal{F}^{-1}(f_{k_1})\|_{L^4} \cdot \|\mathcal{F}^{-1}(\mu_2 \cdot f_{k_2})\|_{L^4} \\ & \leq C 2^{k_1} \|f_{k_1}\|_{V_{k_1}} \cdot 2^{-k_2} \|f_{k_2}\|_{V_{k_2}} + C 2^{-k_1} \|f_{k_1}\|_{V_{k_1}} \cdot 2^{k_2} \|f_{k_2}\|_{V_{k_2}}. \end{aligned}$$

This completes the proof of (10.2).

We show now that

$$\left[\sum_{k \in \mathbb{Z}} \left\| 2^k \chi_k(\xi) (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * f_{k_2}) \right\|_{W_k}^2 \right]^{1/2} \leq C \|f_{k_1}\|_{V_{k_1}} \cdot \|f_{k_2}\|_{V_{k_2} \cap W_{k_2}}. \quad (10.4)$$

In view of the definition (2.7), the left-hand side of (10.4) is bounded by

$$\begin{aligned} & C \left[\sum_{k \in \mathbb{Z}} \left\| 2^k \chi_k(\xi) \cdot (\mu/\xi) (\tau - \omega(\xi, \mu) + i)^{-2} \cdot (f_{k_1} * f_{k_2}) \right\|_{X_k + Y_k}^2 \right]^{1/2} \\ & + C \left[\sum_{k \in \mathbb{Z}} \left\| 2^k \chi_k(\xi) (\tau - \omega(\xi, \mu) + i)^{-1} \cdot (f_{k_1} * (\partial_\mu + I) f_{k_2}) \right\|_{X_k + Y_k}^2 \right]^{1/2}. \end{aligned} \quad (10.5)$$

The first term in (10.5) is dominated by the left-hand side of (10.2), which suffices.

Using (2.4) and Lemma 5.5, the second term in (10.5) is bounded by

$$\begin{aligned} & C 2^{k_2} \|f_{k_1} * (\partial_\mu + I) f_{k_2}\|_{L^2} \leq C \|\mathcal{F}^{-1}(f_{k_1})\|_{L^4} \cdot 2^{k_2} \|\mathcal{F}^{-1}((\partial_\mu + I) f_{k_2})\|_{L^4} \\ & \leq C \|f_{k_1}\|_{V_{k_1}} \cdot \|(\partial_\mu + I) f_{k_2}\|_{X_{k_2} + Y_{k_2}}. \end{aligned}$$

This completes the proof of (10.4).

The proposition follows from the estimates (10.2) and (10.4). \square

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UNIVERSITY OF TORONTO

E-mail address: colliand@math.toronto.edu

UNIVERSITY OF WISCONSIN–MADISON

E-mail address: ionescu@math.wisc.edu

UNIVERSITY OF CHICAGO

E-mail address: cek@math.uchicago.edu

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

E-mail address: gigliola@math.mit.edu